1.(a) Let $p_0$ be the momentum of one of the spheres before collision. Since the total momentum is zero in the center of mass frame the other sphere must have momentum $-p_0$ before collision. The momenta of the spheres after the collision are $p_1$ and $-p_1$. The collision is elastic, so kinetic energy is conserved. Since the spheres have identical masses this implies $|p_0| = |p_1|$. The collision is sketched in the figure below where the spheres have radius $a$ and collide with impact parameter $b$. The scattering angle, the angle between $p_0$ and $p_1$ is $\theta'$. The key point is that the spheres cannot exert any force on one another in the direction perpendicular to the line joining their centers, and therefore the momentum component in this direction is conserved for each sphere separately. From the diagram we see that $\theta'$ and $\phi'$ are related by $\theta' + 2\phi' = \pi$. In addition, from the right triangle whose hypotenuse is the line joining the centers of the spheres, we see $\sin \phi' = b/2a$. Combining these two equations gives us a relation between the impact parameter and the scattering angle:

$$\frac{b}{2a} = \sin \phi' = \sin(\pi/2 - \theta'/2) = \cos(\theta'/2) = \sqrt{\frac{1 + \cos \theta'}{2}} \Rightarrow \cos \theta' = \frac{b^2}{2a^2} - 1.$$

If we differentiate this expression with respect to $\theta'$ we find

$$-\sin \theta' = \frac{b}{a^2} \frac{db}{d\theta'} \Rightarrow \left| \frac{db}{d\theta'} \right| = \frac{a^2}{b} \sin \theta'.$$

The differential cross section is then

$$d\sigma = b db d\phi' = b \left| \frac{db}{d\theta'} \right| d\theta' d\phi' = a^2 \sin \theta' d\theta' d\phi' = a^2 d\Omega' \Rightarrow \frac{d\sigma}{d\Omega'} = a^2.$$

![FIG. 1. Elastic scattering of hard spheres of radius $a$ in the center of mass frame.](image)

(b) From part (a), the scattering angle in the center of mass frame is related to the impact parameter by $\cos(\theta'/2) = \frac{b}{2a}$. From the lecture notes (pg. 21-5), the scattering angle in the lab frame is given by $\cos \theta = \frac{b}{2a}$, and thus $\cos(\theta'/2) = \cos \theta$. Taking differentials we find $\sin(\theta'/2)d\theta' = 2\sin \theta d\theta$. Taking the product of these two equations gives

$$\sin(\theta'/2) \cos(\theta'/2)d\theta' = 2 \sin \theta \cos \theta d\theta \Rightarrow \sin \theta' d\theta' = 4 \sin \theta \cos \theta d\theta \Rightarrow d\Omega' = \sin \theta' d\theta' d\phi' = 4 \sin \theta \cos \theta d\theta d\phi' = 4 \cos \theta d\Omega,$$

where we have used the fact that the scattering plane is the same in the center of mass and lab frames so $d\phi = d\phi'$. The differential scattering cross section in the lab frame is therefore

$$\frac{d\sigma}{d\Omega} = 4a^2 \cos \theta,$$

in agreement with the result in the lecture notes.
2. (a) Consider a volume $V$ bounded by a closed surface $S = \partial V$. The number of particles in this volume at time $t$ is

$$N(t) = \int_V dV \, n(x, t),$$

where $n(x, t)$ is the number density of particles at point $x$ at time $t$. $N$ can only change in time if particles are flowing into or out of $V$ through the boundary, hence

$$\frac{\partial N}{\partial t} = \int_V dV \frac{\partial n(x, t)}{\partial t} = -\int_S ds \cdot J_n(x, t) = -\int_V dV \nabla \cdot J_n(x, t).$$

Note that $ds$ is the outward-directed differential surface element, since if $J_n$ is pointing out of the volume $V$ the number of particles in $V$ is decreasing. In the last step we have employed the divergence theorem to convert the surface integral to a volume integral. From this equality we have

$$\int_V dV \left( \frac{\partial n(x, t)}{\partial t} + \nabla \cdot J_n(x, t) \right) = 0 \quad \Rightarrow \quad \frac{\partial n}{\partial t} + \nabla \cdot J_n = 0,$$

where the last step follows because our choice of the volume $V$ was arbitrary.

(b) Taking the divergence of the transport equation, we find:

$$\nabla \cdot J_n = -D \nabla \cdot \nabla n = -D \nabla^2 n.$$

Using this to eliminate $\nabla \cdot J_n$ in the continuity equation gives us a single partial differential equation for $n(x, t)$:

$$\frac{\partial n}{\partial t} - D \nabla^2 n = 0.$$

(c) Replacing the spatial derivatives in the diffusion equation with $1/L$, and the time derivative with $1/T$ gives $n/T - Dn/L^2 \sim 0$, which says that the length and time scales are related by $T \sim L^2/D$. We take the diffusivity to be $D = c \ell/3$, where $c = (8\pi/\pi M)^{1/2}$, and the mean free path is $l = 1/n\pi d^2$. The characteristic time $T$ for the ammonia molecules to diffuse a distance $L$ is then

$$T \approx \frac{L^2}{D} = \frac{3L^2}{c \ell} = 3\pi L^2 n_d^2 \sqrt{\frac{\pi M}{8\tau}} = 3pL^2 d^2 \sqrt{\frac{M\pi^3}{8\tau^3}},$$

where in the last step we used the ideal gas law $n = N/V = p/\tau$. The distance between the front of the room and the back is $L \approx 500$ cm, and we also know $p = 1.01 \cdot 10^6$ dyne cm$^{-2}$, $d \approx 10^{-8}$ cm, $T = 300$ K, $M = 17n_p$. Numerically,

$$T = 3(1.01 \cdot 10^6 \text{ dyne cm}^{-2})(500 \text{ cm})^2(10^{-8} \text{ cm})^2 \sqrt{\frac{17(1.67 \cdot 10^{-24} \text{ g})\pi^3}{8[(1.38 \cdot 10^{-16} \text{ erg K}^{-1})(300 \text{ K})]^3}} \approx 10^5 \text{ s} \approx 28 \text{ hr}.$$

3. For a classical gas of particles of charge $q$, the conductivity is given by Eq. (93) on pg. 413 of K&K: $\sigma = nq^2 \tau_c/m$. From Eq. (31) on pg. 402 of K&K, the thermal conductivity is $K = D \tilde{C}_V$, where $D$ is the diffusivity and $\tilde{C}_V$ is the heat capacity per unit volume, which for a classical gas is $\tilde{C}_V = 3n/2$. Therefore the desired ratio is

$$\frac{K}{\tau \sigma} = \frac{D \tilde{C}_V}{\tau \sigma} = \frac{D}{\tau} \frac{3n}{2} \frac{m}{nq^2 \tau_c} = \frac{3D}{2q^2 \tau \tau_c}.$$

The diffusivity can be written as $D = c \ell/3$, where $\bar{c}$ is the mean speed and $\ell$ the mean free path. We can relate the mean free path to the relaxation time by $\ell = \bar{c} \tau_c$. Therefore $D = \bar{c}^2 \tau_c/3$, and if we replace the mean speed squared with the square of the rms speed $\bar{c}^2 \approx v_{\text{rms}}^2 = 3\tau/m$, we find $D = \tau \tau_c/m$ and thus

$$\frac{K}{\tau \sigma} = \frac{3}{2q^2}, \quad \frac{K}{\tau \sigma} = \frac{3k_B^2}{2q^2},$$

since in conventional units we use $\tau = k_B T$, and $K \to K/k_B$. 
4. (a) The conduction electrons in a metal are a cold Fermi gas. The heat capacity of a cold Fermi gas is given in conventional units in Eq. (38) on page 193 of K&K: \( C = \pi^2 N k_B^2 T / 2 \varepsilon_F \). If we divide this by the volume \( V \) to get the heat capacity per unit volume, and use \( N/V = n \), where \( n \) is the conduction electron density, and \( \varepsilon_F = (h^2/2m)(3\pi^2n)^{2/3} \), we find

\[
\dot{C}_{el} = \frac{\pi^2 n k_B^2 T}{2} \frac{2m}{h^2(3\pi^2n)^{2/3}} = \frac{mk_B^2 T}{h^2} \left( \frac{n\pi^2}{9} \right)^{1/3}
\]

\[
= \left( \frac{9.11 \cdot 10^{-28}}{(1.38 \cdot 10^{-16} \text{erg K}^{-1})^2 (300 \text{K})} \right) \left( \frac{(8 \cdot 10^{22} \text{cm}^{-3})^2}{9} \right)^{1/3}
\]

\[
= 2.1 \cdot 10^5 \text{erg K}^{-1} \text{cm}^{-3}
\]

(b) The thermal conductivity is related to the diffusivity and heat capacity per unit volume via \( K = D \dot{C}_{el} \). From Eq. (92) on pg. 413 of K&K, the diffusivity of a Fermi gas is \( D = v_F^2 \tau_c/3 \), where \( v_F \) is the Fermi velocity and the relaxation time can be written \( \tau_c = l/v_F \), where \( l \) is the mean free path. Since \( \varepsilon_F = m v_F^2 / 2 \), we can use the expression above for the Fermi energy to find the Fermi velocity: \( v_F = (h/m)(3\pi^2 n)^{1/3} \). Combining these results with the result from (a), we find

\[
K = D \dot{C}_{el} = \frac{l}{3m} \left( \frac{n\pi^2}{9} \right)^{1/3} \frac{mk_B^2 T}{h^2} \left( \frac{\pi^2 n}{9} \right)^{2/3}
\]

\[
= \left( \frac{400 \cdot 10^{-8} \text{cm}}{(1.38 \cdot 10^{-16} \text{erg K}^{-1})^2 (300 \text{K})} \right) \left( \frac{\pi^2 (8 \cdot 10^{22} \text{cm}^{-3})}{9} \right)^{2/3}
\]

\[
= 4.3 \cdot 10^7 \text{erg cm}^{-1} \text{s}^{-1} \text{K}^{-1}
\]

(c) The electrical conductivity of a Fermi gas is given in Eq. (94) on pg. 413 of K&K: \( \sigma = nq^2 \tau_c/m \). Using \( \tau_c = l/v_F \) and the expression for \( v_F \) from part (b) we find

\[
\sigma = \frac{nq^2 l}{mv_F} = \frac{n^{2/3} q^3 l}{h(3\pi^2)^{1/3}} = \left( \frac{8 \cdot 10^{22} \text{cm}^{-3}}{(1.602 \cdot 10^{-19} \text{C})^2 (400 \cdot 10^{-8} \text{cm})} \right) \left( \frac{10^7 \text{erg}}{1 \text{J}} \right)
\]

\[
= 5.9 \cdot 10^5 \text{ohm}^{-1} \text{cm}^{-1}
\]

5. (a) The Boltzmann transport equation in the relaxation time approximation is

\[
v_x \frac{df}{dx} = -\frac{1}{\tau_c} (f - f_0)
\]

where we have dropped the time derivative and acceleration terms because we have steady-state conditions, and we have assumed the distribution function depends only on \( x \). Writing \( f = f_0 + f_1 \), where \( f_1 \) is a small correction to the equilibrium distribution, \( f_0 \), we find from substituting this into the Boltzmann equation

\[
f_1 \approx -\tau_c v_x \frac{d}{dx} f_0 = -\tau_c v_x \frac{d}{dx} \exp[(\mu - \varepsilon)/\tau] = -\tau_c v_x f_0 \left[ \frac{d}{dx} \left( \frac{\mu}{\tau} \right) + \frac{\varepsilon}{\tau^2} \frac{d\tau}{dx} \right],
\]

where we have used \( f_0 = \exp[(\mu - \varepsilon)/\tau] \). We are told that the temperature is a function of \( x \) and the particle concentration is constant. We can use this last fact to find the \( x \)-dependence of \( \mu/\tau \). The particle concentration at the point \( x \) can be written:

\[
n(x) = \int d\varepsilon D(\varepsilon) f_0(\varepsilon) = e^{\mu(x)/\tau(x)} \int d\varepsilon A e^{\varepsilon/\tau(x)} = e^{\mu(x)/\tau(x)}[\tau(x)]^{3/2} A \int dy y^{1/2} e^{-y},
\]

where in the first step we used the fact that the density of states as a function of energy for non-relativistic particles in three dimensions is \( D(\varepsilon) = A \varepsilon^{1/2} \) where \( A \) is a collection of constants. In the second step we introduced the dimensionless variable \( y = \varepsilon/\tau(x) \). Differentiating the last equality with respect to \( x \) and using \( dn/dx = 0 \), we find:

\[
e^{\mu/\tau^{3/2}} \frac{d}{dx} \left( \frac{\mu}{\tau} \right) + e^{\mu/\tau} \frac{3}{2} \tau^{1/2} \frac{d\tau}{dx} = 0 \Rightarrow \frac{d}{dx} \left( \frac{\mu}{\tau} \right) = -\frac{3}{2\tau} \frac{d\tau}{dx}.
\]
Using this expression in the equation for \( f_1 \) gives
\[
f \approx f_0 - v_x \tau_c \left( -\frac{3}{2\tau} + \frac{\varepsilon}{\tau^2} \right) f_0 \frac{\partial \tau}{\partial x}.
\]

(b) The energy flux in the \( x \)-direction is
\[
J_u = \int d\varepsilon v_x f_0(D(\varepsilon)) \approx \int d\varepsilon v_x(f_0 + f_1)D(\varepsilon) = -\left( \frac{\partial \tau}{\partial x} \right) \tau_c \int d\varepsilon v_x^2 \left( \frac{3\varepsilon}{2\tau} + \frac{\varepsilon^2}{\tau^2} \right) f_0(D(\varepsilon)),
\]
where in the last step we have used the fact that the term involving \( f_0 \) vanishes because there is no energy flux in equilibrium, and we have substituted the expression for \( f_1 \) from part (a). Since \( \varepsilon = (m/2)(v_x^2 + v_y^2 + v_z^2) \), and in equilibrium the system is isotropic, we can replace \( v_x^2 \) by \( 2\varepsilon/3m \).

(c) To evaluate the integral note
\[
\int d\varepsilon \int \varepsilon^k f_0(D(\varepsilon)) = \langle \varepsilon^k \rangle \Rightarrow \int d\varepsilon \varepsilon^k f_0(D(\varepsilon)) = n \langle \varepsilon^k \rangle,
\]
where \( n \) is the (uniform) density. Therefore we need to evaluate the thermal average of powers of the energy. Since \( \varepsilon = mv^2/2 \), and we know the speed distribution for a classical gas is given by the Maxwell distribution: \( p(v) = Cv^2 e^{-\alpha v^2} \), where \( \alpha = m/2\tau \), and \( C = 4\pi (m/2\tau)^{3/2} = (4/\sqrt{\pi})\alpha^{3/2} \), we can write
\[
\langle \varepsilon^k \rangle = \left( \frac{m}{2} \right)^k \int_0^\infty dv C v^{2k+1} e^{-\alpha v^2} = \left( \frac{m}{2} \right)^k \frac{2}{\sqrt{\pi} \alpha^k} \Gamma[(2k+3)/2]
\]
\[
= \frac{2}{\sqrt{\pi}} \tau^k ((2k+1)/2)((2k-1)/2)\ldots(3/2)(1/2)\Gamma(1/2) = \frac{\varepsilon^k}{2k}(2k+1)!!.
\]
See the solution to Homework Set 2, Problem 1 for more details on the evaluation of the integral. In particular:
\[
\langle \varepsilon^2 \rangle = \frac{\tau^2}{4} (5 \cdot 3 \cdot 1) = \frac{15}{4} \tau^2, \quad \langle \varepsilon^3 \rangle = \frac{\tau^3}{8} (7 \cdot 5 \cdot 3 \cdot 1) = \frac{105}{8} \tau^3.
\]
Using these results, the integral we want is
\[
\int d\varepsilon v_x^2 \left( -\frac{3\varepsilon}{2\tau} + \frac{\varepsilon^2}{\tau^2} \right) f_0(D(\varepsilon)) = -\frac{1}{m\tau} n \langle \varepsilon^2 \rangle + \frac{2}{3m\tau^2} n \langle \varepsilon^3 \rangle = -\frac{15}{4} n\tau^2 + \frac{35}{4} n\tau^3 = \frac{5n\tau^3}{m},
\]
and the energy flux in the \( x \)-direction is
\[
J_u = -\left( \frac{d\tau}{dx} \right) \tau_c \frac{5n\tau^3}{m} = -K \frac{d\tau}{dx},
\]
from which we find
\[
K = \frac{5n\tau^3}{m}.
\]

6. We have a fluid flowing through a circular cylinder of length \( L \) and radius \( a \). Consider a cylindrical region of the fluid with radius \( r < a \), whose axis coincides with the axis of the tube. For steady-state flow the total force on this cylinder of fluid must vanish. Consider the forces on the cylinder along the axis of the tube. There is a force \( p\pi r^2 \) from the pressure difference, \( p \), between the end faces of the cylinder, and a force \( \eta(dv/dr)A \) due to viscous drag, where \( A = 2\pi r L \) is the surface area of the cylinder. Hence
\[
p\pi r^2 + 2\pi r L \eta \frac{dv}{dr} = 0 \quad \Rightarrow \quad \frac{dv}{dr} = -\frac{p}{2\pi \eta} r.
\]
This is a differential equation for the velocity as a function of radius, which can be immediately integrated to give \( v(r) = -\frac{pr^2}{4L\eta} + C \), where \( C \) is a constant of integration. We have the boundary condition \( v(a) = 0 \), from which we can evaluate \( C \), and thus find

\[
v(r) = \frac{p}{4L\eta}(a^2 - r^2).
\]

To find the volume flow rate we note that the fluid between \( r \) and \( r + dr \) has area \( 2\pi r dr \) and flows with velocity \( v(r) \), hence:

\[
\dot{V} = \int_0^a dr \, 2\pi rv(r) = \frac{\pi p}{2L\eta} \int_0^a dr \, r(a^2 - r^2) = \frac{\pi pa^4}{8L\eta}.
\]