1. We are considering a completely degenerate (i.e. \( \tau = 0 \)) ultra-relativistic Fermi gas. We shall assume the particles have spin-1/2, since we want to apply the results to the electrons in a neutron star. The relationship between energy and momentum is \( \varepsilon = pc \). We shall first work out the density of states in energy, \( D(\varepsilon) \). Since we are still considering single-particle states in a box of volume \( V \), the number of states with a momentum vector in the element \( d^3p \) is still given by

\[
    dn(p) = \frac{2V}{(2\pi)^3 \hbar^3} d^3p
\]

In spherical coordinates the element \( d^3p = p^2 dp d\Omega \), and integrating over the direction of the momentum vector gives \( d^3p = 4\pi p^2 dp \). Changing the independent variable from \( p \) to \( \varepsilon \) using \( \varepsilon = pc \) gives

\[
    dn(\varepsilon) \equiv D(\varepsilon) d\varepsilon = \frac{V}{\pi^2 (\hbar c)^3} \varepsilon^2 d\varepsilon.
\]

The Fermi energy is defined via

\[
    N = \int_0^{\varepsilon_F} d\varepsilon D(\varepsilon) = \frac{V}{\pi^2 (\hbar c)^3} \frac{\varepsilon_F^4}{4} \Rightarrow 
    \varepsilon_F = \pi \hbar c \left( \frac{3n}{\alpha} \right)^{1/3}.
\]

The ground state energy is

\[
    U_0 = \int_0^{\varepsilon_F} d\varepsilon \varepsilon D(\varepsilon) = \frac{V}{\pi^2 (\hbar c)^3} \frac{\varepsilon_F^4}{4} = \frac{V}{\pi^2 (\hbar c)^3} \frac{\varepsilon_F^3}{3} \times \frac{3}{4} \varepsilon_F = \frac{3}{4} N \varepsilon_F.
\]

Next we reconsider the question of the electron and proton concentrations in a neutron star, following the discussion on pages 12-4 through 12-6 of the lecture notes. We shall now treat the electrons as an ultra-relativistic degenerate Fermi gas, while continuing to assume the protons and neutrons are well described by the non-relativistic formulas since \( m_p, m_n \gg m_e \). Using our above result, the Fermi energy for the electrons can be written

\[
    \varepsilon_{F,e} = \pi \hbar c \left( \frac{3n_e}{\alpha} \right)^{1/3} = \varepsilon_{F,0} \left( \frac{n}{n_0} \right)^{1/3} x^{1/3} \alpha,
\]

where \( \varepsilon_{F,0} = \frac{(\hbar^2/2m_e)(3\pi^2n_0)^{2/3}}{2(1.67 \cdot 10^{-24} g)(3.00 \cdot 10^8 \text{ m/s})} \left( \frac{1.05 \cdot 10^{-27} \text{ erg s}}{3\pi^2(2.9 \cdot 10^{38} \text{ cm}^{-3})} \right)^{1/3} = 4.67, \)

is a dimensionless combination of constants. Using this expression in the energy equation \( \varepsilon_{F,n} = \varepsilon_{F,p} + \varepsilon_{F,e} - E \), where \( E = (m_n - m_p - m_e)c^2 \) we have

\[
    \varepsilon_{F,0} \left( \frac{n}{n_0} \right)^{2/3} (1 - x)^{2/3} = \varepsilon_{F,0} \left( \frac{n}{n_0} \right)^{2/3} \frac{m_n}{m_p} x^{2/3} + \frac{1}{3} \alpha x^{1/3} \left( \frac{n_0}{n} \right)^{2/3} \frac{E}{\varepsilon_{F,0}}
\]

\[
    (1 - x)^{2/3} = \frac{m_n}{m_p} x^{2/3} + \left( \frac{n_0}{n} \right)^{1/3} (4.67)x^{1/3} - \left( \frac{n_0}{n} \right)^{2/3} (0.0091).
\]

If we assume \( x \ll 1 \), then we can take \( n_0/n \approx 1 \), \( (1 - x)^{2/3} \approx 1 \), and ignore \( x^{2/3} \) in comparison to \( x^{1/3} \), yielding

\[
    1 \approx 4.67 x^{1/3} - 0.0091 \Rightarrow x \approx 10^{-2}.
\]
If we instead only use the approximation \( n_0/n \approx 1 \) and solve the resulting equation for \( x \) numerically, we find \( x = 8.7 \cdot 10^{-3} \), which is of the same order of magnitude. This result is three orders of magnitude larger than the result obtained in the lecture notes with non-relativistic electrons.

2.(a) For a non-interacting Fermi gas the Fermi energy is

\[
\varepsilon_F = \left( \frac{\hbar^2}{2m_p} \right) \left( \frac{\pi^2 n_p}{m_p} \right)^{2/3} = \frac{(1.05 \cdot 10^{-27} \text{ erg s})^2(\pi^2 \cdot 0.081 \text{ g cm}^{-3})^{2/3}}{6(1.67 \cdot 10^{-24} \text{ g})^{5/3}} = 6.7 \cdot 10^{-16} \text{ erg} = 4.2 \cdot 10^{-4} \text{ eV}.
\]

The Fermi temperature is thus

\[
T_F = \frac{\varepsilon_F}{k_B} = \frac{6.7 \cdot 10^{-16} \text{ erg}}{1.38 \cdot 10^{-16} \text{ erg K}^{-1}} = 4.9 \text{ K},
\]

and the Fermi velocity is

\[
v_F = \sqrt{\frac{2\varepsilon_F}{M}} = \sqrt{\frac{2(6.7 \cdot 10^{-16} \text{ erg})}{3(1.67 \cdot 10^{-24} \text{ g})}} = 1.6 \cdot 10^4 \text{ cm s}^{-1}.
\]

(b) The heat capacity for \( T \ll T_F \) is given by \( C_V = (\pi^2 N/2)(\tau/\tau_F) \). In conventional units this is \( C_V = (\pi^2/2T_F)Nk_B T \). The coefficient of \( Nk_B T \) is therefore \( \pi^2/2T_F = \pi^2/[2(4.9 \text{ K})] = 1.0 \text{ K}^{-1} \). The discrepancy between this result and the experimental value of 2.89 is due to the fact that the interactions between the helium atoms are not negligible under these conditions.

3.(a) As in a previous problem we know the gravitational potential energy should depend on \( G, M, \) and \( R \). The only combination of these quantities which has the dimensions of energy is \( U_g \approx -GM^2/R \).

(b) Since the electrons are a non-relativistic degenerate Fermi gas, the ground state kinetic energy is

\[
U_0 = \frac{3}{5} N \varepsilon_F \approx \frac{\hbar^2}{2m} \left( \frac{3\pi^2 n_e}{2m} \right)^{2/3} \approx \frac{\hbar^2 N^{5/3}}{mV^{2/3}} \approx \frac{\hbar^2}{m} \left( \frac{M}{M_H} \right)^{5/3}
\]

In the first line we substituted the expression for the Fermi energy and began dropping dimensionless constants since we are only interested in the order of magnitude. Note \( N_e, m, \) and \( n_e \) are the electron number, mass, and number density, respectively. In the second line we used \( n_e = N_e/N \) and \( N_e = N \), where \( N \) is the number of protons, which follows from charge neutrality. In the last line we used \( V \propto R^3 \) (up to dimensionless quantities) and \( N = M/M_H \), where \( M \) is the mass of the white dwarf and \( M_H \) is the mass of the proton. This result follows from \( m \ll M_H \), which indicates that the electrons contribute only slightly to the mass of the dwarf.

(c) Setting \( U_0 = -U_g \) and using the results of parts (a) and (b) gives

\[
\frac{GM^2}{R} \approx \frac{\hbar^2 M^{5/3}}{mM_H^{5/3} R^2} \Rightarrow M^{1/3}R = \frac{\hbar^2}{mGM_H^{5/3}} = (1.05 \cdot 10^{-27} \text{ erg s})^2/(9.11 \cdot 10^{-28} \text{ g})(6.6 \cdot 10^{-8} \text{ dyne cm}^2 \text{ g}^{-2})(1.67 \cdot 10^{-24} \text{ g})^{5/3} \approx 10^{20} \text{ g}^{1/3} \text{ cm}
\]

(d) The density of the white dwarf is \( \rho = M/(4\pi R^3/3) \). Using the result of part (c) and \( M = 2 \cdot 10^{33} \text{ g} \) we have

\[
\rho = \frac{M}{\frac{4}{3}\pi R^3} = \frac{M^2}{\frac{4}{3}\pi (M^{1/3}R)^3} = \frac{(2 \cdot 10^{33} \text{ g})^2}{\frac{4}{3}(10^{20} \text{ g}^{1/3} \text{ cm})^3} \approx 10^6 \text{ g cm}^{-3}
\]
(e) For a degenerate gas of neutrons we just replace the electron mass, \( m_e \), in the result from part (b) with \( M_H \), neglecting the small mass difference between a neutron and a proton. Therefore, the ground state kinetic energy of the neutrons is on the order of \( \hbar^2 M^{5/3}/M_H^{8/3} R^2 \). Equating this to the gravitational potential energy in part (a) gives

\[
M^{1/3} R = \frac{\hbar^2}{GM_H^{8/3}} = \frac{(1.05 \cdot 10^{-27} \text{ erg s})^2}{(6.6 \cdot 10^{-8} \text{ dyne cm}^2 \text{ g}^{-2})(1.67 \cdot 10^{-24} \text{ g})^{8/3}} \approx 10^{17} g^{1/3} \text{ cm}.
\]

Thus the radius of a neutron star with a mass equal to that of the sun is \( R = (10^{17} g^{1/3} \text{ cm})/(2 \cdot 10^{33} g)^{1/3} = 8 \cdot 10^5 \text{ cm} = 8 \text{ km} \).

4. For \( \tau < \tau_E \), the number of bosons in the lowest orbital satisfies \( N_0 > 1 \). Hence \( \lambda \) must be very close to unity and therefore we will take \( \lambda = 1 \) throughout the calculation. Since the particles in the condensate have energy zero, the total energy of the Bose gas is

\[
U = \int_0^\infty d\varepsilon \mathcal{D}(\varepsilon) f(\varepsilon, \tau) = \int_0^\infty d\varepsilon \frac{\varepsilon^2}{4\pi^2} \left( \frac{2M}{\hbar^2} \right)^{3/2} \frac{\varepsilon^{1/2}}{\lambda - \varepsilon/\tau - 1}
\approx \int_0^\infty d\varepsilon \frac{\varepsilon^2}{4\pi^2} \left( \frac{2M}{\hbar^2} \right)^{3/2} \frac{\varepsilon^3/2}{\varepsilon/\tau - 1}
= \frac{V}{4\pi^2} \left( \frac{2M}{\hbar^2} \right)^{3/2} \int_0^\infty dx \frac{x^{3/2}}{e^{x} - 1},
\]

where in the last line we changed variables using \( x = \varepsilon/\tau \). Note the definite integral is now just a dimensionless number. The heat capacity is then

\[
C_V = \left( \frac{\partial U}{\partial \tau} \right)_V = \frac{5V}{8\pi^2} \left( \frac{2M}{\hbar^2} \right)^{3/2} \tau^{3/2} \int_0^\infty dx \frac{x^{3/2}}{e^{x} - 1}.
\]

Since \( C_V = \tau \partial (\partial \sigma/\partial \tau)_V \), we can obtain the entropy via the integral

\[
\sigma = \int_0^\tau d\tau' \frac{C_V(\tau')}{\tau'} = \frac{5V}{12\pi^2} \left( \frac{2M}{\hbar^2} \right)^{3/2} \tau^{3/2} \int_0^\infty dx \frac{x^{3/2}}{e^{x} - 1} \int_0^\tau d\tau' (\tau')^{1/2}
= \frac{5V}{12\pi^2} \left( \frac{2M}{\hbar^2} \right)^{3/2} \tau^{3/2} \int_0^\infty dx \frac{x^{3/2}}{e^{x} - 1}.
\]

For \( \tau > \tau_E \) we cannot use \( \lambda \approx 1 \), and we would have to keep \( \lambda = e^{\mu/\tau} \) explicitly in the expression for \( U \). Since we will have to take a derivative with respect to \( \tau \) of this expression to get \( C_V \), we have to take into account the fact that at fixed \( N \), \( \mu \) is a function of \( \tau \). To do this we would first use \( N = \int d\varepsilon \mathcal{D}(\varepsilon) f(\varepsilon, \tau, \mu) \) to find \( N \) as a function of \( \mu \) and \( \tau \). We could then invert this relation to give \( \mu(N, \tau) \), and use this to eliminate \( \mu \) in favor of \( N \) in the integral expression for \( U \). Once we have found \( U(N, V, \tau) \), we could find \( C_V \) and \( \sigma \) as above.

5. For a single orbital of a boson system we have \( \langle N \rangle = [e^{(e-\mu)/\tau} - 1]^{-1} \). From Eq. (5.83) in K&K, which we derived on a previous problem set, we know \( \langle (\Delta N)^2 \rangle = \tau (\partial (\langle N \rangle)/\partial \mu) \). Thus

\[
\langle (\Delta N)^2 \rangle = \tau \left( \frac{\partial \langle N \rangle}{\partial \mu} \right) = \tau \left( \frac{\partial}{\partial \mu} \left( \frac{1}{e^{(e-\mu)/\tau} - 1} \right) \right)
= \tau \left( \frac{e^{(e-\mu)/\tau}}{(e^{(e-\mu)/\tau} - 1)^2} \right)
= \frac{1}{e^{(e-\mu)/\tau} - 1} + \frac{1}{(e^{(e-\mu)/\tau} - 1)^2}
= \langle N \rangle + \langle N \rangle^2 = \langle N \rangle (\langle N \rangle + 1).
\]

6. The Einstein condensation temperature is related to the mass \( M \) of the bosons and their concentration \( n = N/V \) via \( \tau_E = (2\pi\hbar^2/M)(n/2.612)^{2/3} \). Solving this equation for the concentration and using \( M = 87m_p \), where \( m_p \) is the
proton mass, and the value of $\tau_E$ given in the problem we have

$$n = 2.612 \left[ \frac{\tau_E M}{2\pi \hbar^2} \right]^{3/2} = 2.612 \left[ \frac{(1.38 \cdot 10^{-16} \text{erg K}^{-1})(170 \cdot 10^{-9} \text{K})(87)(1.67 \cdot 10^{-24} \text{g})}{2\pi(1.05 \cdot 10^{-27} \text{erg s})^2} \right]^{3/2} = 2.9 \cdot 10^{13} \text{cm}^{-3}.$$  

The mass density is

$$\rho = n M = (2.9 \cdot 10^{13} \text{cm}^{-3})(87)(1.67 \cdot 10^{-24} \text{g}) = 4.1 \cdot 10^{-9} \text{g cm}^{-3}.$$  

If we assume the sample is spherical, the volume is given by $V = 4\pi R^3/3$ where $R$ is the radius. Since $n = N/V$, we can write the volume is $V = N/n$, and thus the radius is

$$R = \left( \frac{3V}{4\pi} \right)^{1/3} = \left( \frac{3N}{4\pi n} \right)^{1/3} = \left( \frac{3(2000)}{4\pi(2.9 \cdot 10^{13} \text{cm}^{-3})} \right)^{1/3} = 2.5 \cdot 10^{-4} \text{cm}.$$