1. (a) Consider the momentum flux in the $x$–direction. Let $\theta$ be the angle between the $x$–axis and the direction of the photons velocity. Then the momentum flux in the $x$–direction from photons with frequencies between $\omega$ and $\omega + d\omega$ and directions between $\theta$ and $\theta + d\theta$ is

$$\frac{\delta P_x}{\delta A \delta t} = \left( \frac{u_d \omega}{\hbar \omega} \right) \left( \frac{\hbar \omega}{c} \cos \theta \right) (c \cos \theta) \frac{d\Omega}{4\pi},$$

where $\delta A$ is a differential area perpendicular to the $x$–axis, $\delta t$ is a differential time, and $d\Omega/4\pi = \sin \theta d\theta d\phi/4\pi$ is the (normalized) differential solid angle. Since $u_d \omega d\omega = (h\omega^3/\pi^2 c^3)(d\omega/\exp(h\omega/\tau) - 1)$ is the energy per unit volume per unit range of frequency, and $\hbar \omega$ is the energy per photon, the factor $u_d \omega/\hbar \omega$ is the number density of photons per unit range of frequency. The second factor, $(\hbar \omega/c) \cos \theta$ is the $x$–momentum per photon and the third factor, $c \cos \theta$, is the $x$–component of the photon velocity. The product of these three factors gives the momentum per unit area per unit time carried in the $x$–direction for photons with frequency $\omega$ and direction $\theta$. To obtain the total momentum flux, which is just the pressure, we must integrate over all directions and frequencies. The angular integration gives:

$$\frac{1}{4\pi} \int_0^{2\pi} d\phi \int_0^\pi d\theta \sin \theta \cos^2 \theta = \frac{1}{3},$$

and thus

$$p = \frac{1}{3} \int_0^\infty d\omega u_\omega = \frac{1}{3} u = \frac{\pi^2}{45} \frac{\tau^4}{\hbar^3 c^3},$$

where $u$ is the energy density of the photon gas.

(b) For a cubical box of side $L$, the volume is $V = L^3$ and therefore $dV/V = 3dL/L$. The photon frequencies depend on the length through $\omega_n = c\pi n/L$ and thus $d\omega/\omega = -dL/L$. Finally, we know that the average occupancy of a mode with frequency $\omega$ is $1/\exp(h\omega/\tau) - 1$. At constant entropy this must remain unchanged, which implies $\omega/\tau$ is a constant and hence $d\omega/\omega = d\tau/\tau$. Combining all these results we have at constant entropy:

$$\frac{dV}{V} = 3 \frac{dL}{L} = -\frac{3 d\omega}{\omega} = -\frac{3 d\tau}{\tau} \Rightarrow \left( \frac{\partial \tau}{\partial V} \right) = -\frac{1}{3} \frac{\tau}{V}.$$ 

Using the expression for the total energy of the photon gas $U = (\pi^2/15)(V\tau^4/\hbar^3 c^3)$ we have

$$p = -\left( \frac{\partial U}{\partial V} \right) = -\frac{\pi^2}{15\hbar^3 c^3} \left( \tau^4 \frac{\partial V}{\partial V} + V(4\tau^3) \frac{\partial \tau}{\partial V} \right)$$

$$= -\frac{\pi^2}{15\hbar^3 c^3} \left( \tau^4 - \frac{4}{3} \tau^4 \right)$$

$$= \frac{\pi^2 \tau^4}{45\hbar^3 c^3}.$$ 

This is the same as the result in (a).

2. The average energy in an oscillator with frequency $\omega$ is $\hbar \omega/\exp(h\omega/\tau) - 1$. The energy per photon is just $\hbar \omega$, so the average number of photons in the oscillator is $1/\exp(h\omega/\tau) - 1$. To obtain the total number of photons in the cavity we sum this quantity over all the modes, using $\omega = c\pi n/L$:

$$N = \sum_n \frac{1}{e^{\hbar \omega/\tau} - 1} = 2 \times \frac{1}{8} \times 4\pi \int_0^\infty dn n^2 \frac{\exp(\hbar c n \omega \tau)}{\exp(\hbar c n \omega \tau) - 1}$$

$$= \frac{V}{\pi^2} \left( \frac{\tau}{\hbar} \right)^3 \int_0^\infty dx \frac{x^2}{e^x - 1}.$$ 

The factor of 2 comes from the different polarization states for each frequency, the 1/8 comes from selecting out the octant where $n_x, n_y, n_z$ are all positive and the $4\pi$ comes from integrating over the solid angle in $n$–space. In
the second step we changed variables using \( x \equiv \frac{\hbar \pi cn}{L\tau} \). The integral can be shown to be equal to \( \Gamma(3)\zeta(3) \) where

\[
\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s}.
\]

is the Riemann zeta function (See Lecture 8). Using \( \Gamma(3) = 2! \) and \( \zeta(3) \approx 1.202 \) from a standard table (or Mathematica) we have the total number of photons is

\[
N \approx \frac{2.404 \pi^2 V (\frac{\tau}{\hbar c})^3}{15}.
\]

Combining this with the result for the entropy from the text, \( \sigma = \frac{(4\pi^2 V/45)(\tau/\hbar c)^3}{2} \), we find \( \sigma \approx (3.60)N \).

3. Suppose the hole is in the \( xy \)-plane and \( \theta \) is the angle relative to the \( z \)-axis, which is normal to the hole. The energy per unit volume per unit frequency is \( u_\omega d\omega \), which is defined in the text and in Problem 1. To obtain the flux through an area \( \delta A \) of the hole in a time \( \delta t \) into a solid angle \( \delta \Omega \), we must multiply the energy per unit volume per unit frequency by the fractional solid angle \( \delta \Omega / (4\pi) \), and the volume \( \delta A \times c \cos(\theta) \delta t \), which contains those photons which will pass through the hole in a time \( \delta t \). Therefore, the energy per unit area, per unit time, per unit frequency, per unit solid angle, or the flux density is

\[
\frac{\delta E}{\delta A \delta \omega \delta t \delta \Omega} = \frac{1}{4\pi} u_\omega c \cos \theta \frac{\hbar \omega^3 \cos \theta}{4\pi^3 c^2 (e^{\hbar \omega/\tau} - 1)}.
\]

Integrating this expression over solid angle and frequency,

\[
\frac{\delta E}{\delta A \delta t} = \frac{\hbar}{4\pi^3 c^2} \int_0^{2\pi} d\phi \int_0^{\pi/2} d\theta \sin \theta \cos \theta \int_0^{\infty} d\omega \frac{\omega^3}{(e^{\hbar \omega/\tau} - 1)}
\]

\[
= \frac{\hbar}{4\pi^3 c^2} \times 2\pi \times \frac{1}{2} \times \left( \frac{\tau}{\hbar} \right)^4 \frac{\pi^4}{15}
\]

\[
= \frac{\pi^2}{60\hbar^3 c^2} \tau^4 = \sigma_B T^4,
\]

where \( \sigma_B = \pi^2 k_B^4 / 60\hbar^3 c^2 \) is the Stefan-Boltzmann constant. Note the upper limit of integration on \( \theta \) is \( \pi/2 \) since the cavity occupies the region \( z < 0 \) only.

4. (a) Given \( G, M_\odot, \) and \( R_\odot \), there is a unique combination with the dimensions of energy, so an order of magnitude estimate for the gravitational self energy of the sun is: \( U_g \approx -GM_\odot^2 / R_\odot \approx -4 \times 10^{48} \) ergs.

(b) Equating the total thermal kinetic energy, \( K \), to \( -U_g/2 \) (by the virial theorem) and assuming the particles form a monatomic ideal gas so that \( K = (3/2)Nk_B T \), one finds (using \( N \approx 10^{57} \) as suggested in the book) a temperature \( T \approx 10^7 \) K.

5. (a) Since the modes are independent, the partition function of the photon gas is just the product of the partition function of each mode. For an oscillator with frequency \( \omega_n \) the states are unique labeled by the occupation number \( s = 0, 1, \ldots \) and the energy of state \( s \) is \( \varepsilon_s = s\hbar \omega_n \). Thus the partition function is:

\[
Z = \prod_n \left[ \sum_{s=0}^{\infty} \left( e^{-\hbar \omega_n/\tau} \right)^s \right]
\]

\[
= \prod_n \left[ 1 - \exp(-\hbar \omega_n/\tau) \right]^{-1},
\]

where the index \( n \) runs over the modes.

(b) The free energy is

\[
F = -\tau \log(Z) = \tau \sum_n \log \left[ 1 - \exp(-\hbar \omega_n/\tau) \right].
\]
Transforming the sum to an integral we have

\[ F = \tau \times 2 \times \frac{1}{8} \times 4\pi \int_0^\infty dn n^2 \log \left( 1 - e^{-\hbar \pi cn / L^2} \right) = \frac{V\tau^4}{\pi^2 h^3 c^3} \int_0^\infty dx x^2 \log(1 - e^{-x}). \]

The factor of 2 comes from the different polarization states for each frequency, the 1/8 comes from selecting out the octant where \( n_x, n_y, \) and \( n_z \) are all positive and the 4\( \pi \) comes from integrating over the solid angle in \( n \)-space. In the second step we changed variables using \( x = \hbar \pi cn / L^2 \). Integration by parts gives

\[ \int_0^\infty dx x^2 \log(1 - e^{-x}) = \frac{x^3}{3} \log(1 - e^{-x}) \bigg|_0^\infty - \frac{1}{3} \int_0^\infty \frac{x^3}{e^x - 1} dx = 0 - \frac{1}{3} \pi^4 / 15, \]

where the last integral can be found in the text. Using this result in the above expression for the free energy we find:

\[ F = -\frac{\pi^2 V\tau^4}{45h^3 c^3}. \]

6. Let \( T_m \) be the temperature of the central plane. Once equilibrium has been established this temperature will not change and therefore the total energy flux reaching the center plane must equal the total energy flux out: \( \Phi_{in} = \Phi_{out} \Rightarrow \sigma_B (T_u^4 + T_f^4) = 2\sigma_B T_m^4 \). Solving for \( T_m \) we have

\[ T_m = \left( \frac{T_u^4 + T_f^4}{2} \right)^{1/4}. \]

If we consider the space between the plane at \( T_u \) and the plane at \( T_m \) then the net energy flux density from \( u \) to \( m \) is \( \sigma_B (T_u^4 - T_m^4) \). Using the above result for \( T_m \) we find the net flux is \( (\sigma_B/2)(T_u^4 - T_f^4) \), which is one half of the flux density without the center plane.

7. We know the total phonon energy at temperature \( \tau \) is given by

\[ U = \frac{3V\tau^4}{2\pi^2 h^3 c^3} \int_0^{x_D} dx \frac{x^3}{e^x - 1}, \]

where \( x_D = \theta / T \) and the Debye temperature is \( \theta = (hv/k_B)(6\pi^2 N/V)^{1/3} \). In conventional units the heat capacity at constant volume is \( C_V = (\partial U / \partial T)_V \). At temperatures on the order of \( \theta \) or higher the upper limit of the integral is of order one or smaller and therefore it is useful to expand the integrand as a power series in \( x \):

\[
\begin{align*}
\int_0^{x_D} dx \frac{x^3}{e^x - 1} &= \int_0^{x_D} dx \frac{x^3}{x + x/2! + x^3/3! + O(x^4)} \\
&= \int_0^{x_D} dx \frac{x^2}{1 + x/2! + x^2/3! + O(x^3)} \\
&= \int_0^{x_D} dx x^2 \left[ 1 + (-1) \left( \frac{x}{2!} + \frac{x^3}{3!} + O(x^3) \right) + \frac{(-1)(-2)}{2!} \left( \frac{x}{2!} + \frac{x^3}{3!} + O(x^3) \right)^2 + O(x^3) \right] \\
&= \int_0^{x_D} dx x^2 \left[ 1 - \frac{x}{2} + \frac{x^2}{12} + O(x^3) \right] \\
&= \frac{x_D^3}{3} - \frac{x_D^5}{8} + \frac{x_D^7}{60} + O(x_D^9),
\end{align*}
\]

where we have used the expansion \((1 + y)^n = 1 + ny + [n(n - 1)/2!]y^2 + O(y^3)\). Using this expression for the integral in the equation for the total energy we find

\[ U = 3Nk_B T - \frac{9Nk_B}{8} \theta + \frac{3Nk_B \theta^2}{20} T + O \left( \frac{1}{T^2} \right). \]
Note that it can be shown that the temperature independent term is exactly minus the zero-point energy of the oscillators. Differentiating we find

\[ C_V = 3 N k_B \left( 1 - \frac{1}{20} \left( \frac{\theta}{T} \right)^2 + \mathcal{O} \left( \frac{1}{T^3} \right) \right). \]

We see that in the limit of very high temperatures \( T \gg \theta \) we recover the classical result \( C_V = 3 N k_B \). To check the \( \mathcal{O}(T^{-2}) \) term note that to this order \( C_V(T = \theta) = (19/20)3 N k_B = (19/20)C_V(T = \infty) \), which agrees with the data in Table 4.2 of K&K.

8.(a) Since there are no transverse sound waves in the liquid, the equation relating the Debye number to the total number of atoms is

\[ \frac{1}{8} \times 4 \pi \int_0^{n_D} d n n^2 = 3 N, \]

which gives \( n_D = (18N/\pi)^{1/3} \). The Debye temperature is then

\[ \theta = \frac{\pi \hbar \nu n_D}{k_B L} = \frac{\hbar \nu}{k_B} \left( \frac{18 \pi^2 N}{V} \right)^{1/3} = 28.6 \text{ K}. \]

(b) The equation for the low temperature heat capacity, when written in terms of \( \theta \), is unchanged in the absence of transverse modes. In conventional units we have \( C_V = (12 \pi^4 N k_B/5)(T/\theta)^3 \). To get the heat capacity per unit mass of liquid we divide this by the mass of the sample which is \( M = N \, m \), where \( m \) is the mass of a \(^4\text{He} \) atom. The heat capacity in J g\(^{-1}\)K\(^{-1}\) is then

\[ \frac{C_V}{M} = \frac{12 \pi^4 k_B}{5 m \theta^3} T^3 = 0.0208 \times T^3, \]

in good agreement with the experimental value given in the text.