Homework 11 - Solutions

4th January 2005

Problem 1

In steady state situation, in the boltzman equation $\alpha = 0$ and $\partial f/\partial t = 0$ so the boltzman equation gives

$$-\frac{f - f_0}{\tau_c} = v_x \frac{\partial f}{\partial x}$$

To the lowest order

$$f = f_0 - \tau_c v_x \frac{\partial f_0}{\partial x}$$

We know

$$f_0 = c e^{(\mu - \epsilon)/\tau}$$

$$\epsilon = \frac{1}{2} m v^2$$

$$\frac{\mu}{\tau} = \log\left(\frac{n}{n_Q}\right) = \log(n\tau^{-3/2}) + const$$

Assuming $n =$ const as asked

$$f = f_0 - \tau_c v_x f_0 \left(\frac{\partial}{\partial x}\frac{\mu}{\tau} - \epsilon \frac{\partial}{\partial x}\frac{1}{\tau}\right) = \tau_c v_x \left(-\frac{3}{2\tau} + \frac{\epsilon}{\tau^2}\right) f_0 \frac{\partial \tau}{\partial x}$$

The energy current is

$$\begin{aligned}
J_u &= \int f v_x \epsilon D(\epsilon) d\epsilon \\
&= -\tau_c \frac{\partial \tau}{\partial x} \int v_x^2 \left(-\frac{3\epsilon}{2\tau} + \frac{\epsilon^2}{\tau^2} \right) f_0 D(\epsilon) d\epsilon \\
&= -\tau_c \frac{\partial \tau}{\partial x} \int v_x^2 \left(-\frac{3mv^2}{4\tau} + \frac{mv^4}{4\tau^2} \right) f_0 D(\epsilon) d\epsilon \\
&= -\tau_c \frac{\partial \tau}{\partial x} \left(\frac{3m}{4\tau} < v_x^2 v^2 > + \frac{m^2}{4\tau^2} < v_x^2 v^4 > \right)
\end{aligned}$$

Now we evaluate the expectation values. By symmetry

$$< v_x^2 v^2 > = \frac{1}{3} < v^4 >$$

$$< v_x^2 v^4 > = \frac{1}{3} < v^6 >$$

We have the following relations

$$\int e^{-av^2} d^3v = \left(\int e^{-av_x^2} dv_x \right)^3 = \left(\frac{\pi}{a} \right)^3$$

$$\int v^2 e^{-av^2} d^3v = -\frac{\partial}{\partial a} \int e^{-av^2} d^3v = \frac{3}{2a} \left(\frac{\pi}{a} \right)^3$$

$$\int v^4 e^{-av^2} d^3v = \frac{3 \cdot 5}{2^2 a^2} \left(\frac{\pi}{a} \right)^3$$

$$\int v^6 e^{-av^2} d^3v = \frac{3 \cdot 5 \cdot 7}{2^3 a^3} \left(\frac{\pi}{a} \right)^3$$

And the expectation values are

$$< v^2 > = \frac{3}{2a}$$

$$< v^4 > = \frac{3 \cdot 5}{2^2 a^2}$$

$$< v^6 > = \frac{3 \cdot 5 \cdot 7}{2^3 a^3}$$

and in our case

$$a = \frac{m}{2\tau}$$

Pluging back these formulas into the furmula for the energy current we get

$$J_u = -5\frac{\tau_c \tau}{m}\frac{\partial \tau}{\partial x}$$

Problem 2

These are the data for carbon (diamond)
$\rho = 2267$ kg/m^3
$v = 18350$m/s
$K = 140$W/mK
$C_p = 6.115$ J/K mol
$M_m = 12 \times 10^{-3}$kg/mol
The two ways to get diffusivity are

$$D_1 = K/C_p^{(V)}$$

$$D_2 = vl_0$$

where $C_p^{(V)}$ is the heat capacity per volume at constant pressure and l_0 is the distance between atoms. We get

$$D_1 = \frac{K}{C_p^{(V)}} = \frac{K\rho}{C_p M_m} \approx 1.2 \times 10^{-4}\mathrm{m}^2/\mathrm{s}$$

$$l_0 = V_0^{-1/3} = (m/\rho)^{-1/3} = (M_m/N_a\rho)^{-1/3} \approx 1.64 \times 10^{-10}\mathrm{m}$$
$$D_2 = vl_0 \approx 3 \times 10^{-6}\mathrm{m}^2/\mathrm{s}$$

For selenium we have
C_p =25.36 J/mol K
K =0.52 W/m K
ρ =4819 kg/m^3
M =0.079 kg/mol
c =3350 m/s
from which we slimilarly get
$D_1 \approx 3.3 \times 10^{-7}\mathrm{m}^2/\mathrm{s}$
$l \approx 3 \times 10^{-10}\mathrm{m}$
$D_2 \approx 10^{-6}\mathrm{m}^2/\mathrm{s}$

Problem 3

The variation of the temperature $T(t, x)$ on the surface of the earth is

$$T(t, 0) = T_1 + T_0(\cos(\omega_d t) + \cos(\omega_a t))$$

where $T_1 = 283\mathrm{K}$, $T_0 = 10\mathrm{K}$ and ω_d and ω_a are the daily and annual "angular" frequencies.

The temperature in the earth satisfies the diffusion equation

$$\frac{\partial T}{\partial t} = -D\frac{\partial^2 T}{\partial x^2}$$

where x is the depth. At large depth (x large) the temperatre is approximately constant and equals the everage temperature, the T_1. Thus we are solving the diffusion equation subject to bounary conditions and $x = 0$ and $x = \infty$ above.

Since the equation is linear the solution will be of the form

$$T(t, x) = T_1 + T_0 f_d(t, x) + T_0 f_a(t, x)$$

each T_d satisfies the diffusion equation with $f_d(t, 0) = \cos(\omega_d t)$ and $f_d(t, \infty) = 0$ and similarly for f_a.

We try solution of the form

$$f(t, x) = Re(e^{-i\omega t + ax})$$

where Re denotes that we are taking the real part and a is a constant to be determined. Clearly the above solution satisfies the b.c. at $x = 0$ and also at $x = \infty$ if $Re(a) < 0$.

Pluging the above into diffusion equation and canceling the exponentials we get condition

$$-i\omega = -Da^2$$

The solution of this with $Re(a) < 0$ is

$$a = \sqrt{\frac{\omega}{D}}\frac{1}{\sqrt{2}}(-1 - i)$$

so the f is

$$f = Re(e^{-i(\omega t + \sqrt{\frac{\omega}{2D}}x) - \sqrt{\frac{\omega}{2D}}x}) = \cos(\omega t + \sqrt{\frac{\omega}{2D}}x)e^{-\sqrt{\frac{\omega}{2D}}x}$$

Define $k = \sqrt{\frac{\omega}{2D}}$.

Thus the $T(t, x)$ is

$$T(t, x) = T_1 + T_0(\cos(\omega_d t + k_d x)e^{-k_d x} + \cos(\omega_d t + k_a x)e^{-k_a x})$$

We need to make sure that the temperature at the pipe doesn't get below the freezing temperature. The cosines can be at most one so we need to make sure (since T_1 is T_0 above the freezing point)

$$T_0(e^{-k_d x} + e^{-k_a x}) < T_0$$

Pluging the numbers we get that $k_d \approx 19.1$ and $k_a \approx 1.00$ in SI units. The minimum depth of the pipe is then given by the condition

$$e^{-x} + e^{-19.1x} = 1$$

Solving this numericaly gives $x \approx 0.11$m.

Problem 4

Let the slab thickness be L. The $T(t, x)$ satisfies the diffusion equation. Write it as $T(t, x) = T_0 + (T_1 - T_0)f(t, x)$. Then f satisfies the diffusion equation, at $t = 0$ f is 1 inside of the slab, and the boundary is kept at $f = 0$. The diffusion equation is separable so that any solution can be written as a linear combination of the functions of the form $T(t)X(x)$. Clearly from the boundary conditions these functions are

$$T(t) = ce^{-\omega t}$$

$$X(x) = \sin(kx)$$

(we let the slab extend from 0 to L) with $k = n\pi/L$. Pluging this into diffusion equation we get that

$$\omega = D(\pi n/L)^2$$

So the full solution is

$$f = \sum_n C_n e^{-\omega_n t} \sin(kx)$$

The $C's$ are determined from the boundary conditions at $t = 0$ where the function was 1 everywhere, except for the boundary. This gives

$$C = (2/L) \int_0^L dx \sin(n\pi x/L)1 = (1/n\pi)(1 - (-1)^n)$$

Thus the full solution is

$$T(t,x) = T_0 + (T_1 - T_0)\frac{4}{\pi}\sum_k \frac{1}{2k+1} e^{-D(\pi(2k+1)/L)^2 t} \sin\left(\frac{\pi(2k+1)}{L}x\right)$$

Thus each sine decays to zero at rate that is larger for higher frequency. When the f at the center becomes 0.01 of the original value then approximately we can neglect all the other sine curves and the temperature at the center is

$$T_c = T_0 + (T_1 - T_0)\frac{4}{\pi}e^{-D(\pi/L)^2 t}$$

Thus $f = 0.01$ at the center when

$$0.01 = \frac{4}{\pi}e^{-D(\pi/L)^2 t}$$

or $t = 0.491L^2/D = 1.96a^2/D$.

Problem 5

Since $J_u = -K\nabla T$ and in steady state situation $T =$const, pluging this into the modified diffusion equation we get

$$\nabla^2 T = -g_u/K = -G = \text{const}$$

(a) We write this equation in cilindrical coordinates and drop the derivatives with respect to angles (because the situation has cylindrical symmetry). This gives

$$\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial T}{\partial r}\right) = -G$$

multiplying by r and integrating

$$r\frac{\partial T}{\partial r} = -Gr^2/2 + A$$

dividing by r and integrating

$$T = -Gr^2/4 + A\log r + B$$

Since T is regular at the origin, $A = 0$. On the outside the cilinder has temperature T_0 and say it has radius R, we get

$$T = (G/4)(R^2 - r^2)$$

So the raise in the temperature at the center is

$$T_r = GR^2/4$$

(b) In spherical coordinates the diffusion equation becomes

$$\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial T}{\partial r}\right) = -G$$

Following steps analogous to the above we get

$$T = (G/6)(R^2 - r^2)$$

So the raise in the temperature at the center is

$$T_r = GR^2/6$$