

### Homework 9 Solutions

**Problem 1: Kittel 12-1 (a)** The problem of liquefaction is discussed on pages 340-341 of the text. The setup is shown in the figure. Helium flows in the expansion chamber. Some of it liquefies and the rest flows out. The input and output temperature are assumed to be the same. Presumably this latter condition is due to a heat exchanger. The liquefaction coefficient is given by:

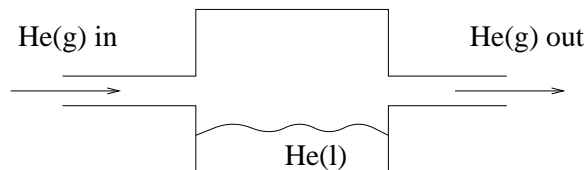


Figure 1: Cartoon of Problem 1.

$$\lambda = \frac{H_{out} - H_{in}}{H_{out} - H_{liq}}$$

We are asked to treat the gaseous helium as a van der Waals gas where the vdW parameters are taken such that:  $2a/b = \tau_{inv}$  and  $2Nb = V_l$ , where  $\tau_{inv}$  is the inversion temperature and  $V_l$  is the volume of one mole of helium *liquid*. The enthalpy of a vdW gas is given by:

$$\begin{aligned} H &= \frac{5}{2}N\tau + \frac{N^2}{V}b\left(\tau - \frac{2a}{b}\right) \\ &= \frac{5}{2}N\tau + N\frac{2Nb}{V}\left(\tau - \tau_{inv}\right) \\ &= \frac{5}{2}N\tau + N\frac{V_l}{V}\left(\tau - \tau_{inv}\right) \\ &= \frac{5}{2}RT + \frac{V_l}{V}R(T - T_{inv}) \end{aligned}$$

where in the last step we put  $N = N_A$  so that we are dealing with one mole of substance. The numerator of  $\lambda$  becomes:

$$\begin{aligned} H_{out} - H_{in} &= RV_l\left(\frac{T_{out} - T_{inv}}{V_{out}} - \frac{T_{in} - T_{inv}}{V_{in}}\right) \\ &= V_l\left(\frac{RT_{in}}{V_{out}} - \frac{RT_{in}}{V_{in}}\right)\left(1 - \frac{T_{inv}}{T_{in}}\right) \\ &\approx V_l\left(1 - \frac{T_{inv}}{T_{in}}\right)(p_{out} - p_{in}) \end{aligned}$$

where we have used  $T_{out} = T_{in}$  and (in the last step) the ideal gas law. The latter approximation is valid if we assume that helium gas only deviates slightly from ideal...an OK assumption away from the boiling point. To determine the denominator of  $\lambda$ , we note that to take a mole of helium from a 4.18K liquid to

a 4.18K gas, we need to add to the system an enthalpy  $\Delta H$  corresponding to the molar heat of vaporization of helium. To then heat up that mole of gas from 4.18K to  $T_{in}$  requires us to add an additional amount of  $(5/2)R(T_{in} - T_{liq})$ . In this latter calculation, we again assumed an ideal gas. Putting it together:

$$\lambda = \frac{V_l(1 - \frac{T_{inv}}{T_{in}})(p_{out} - p_{in})}{\Delta H + \frac{5}{2}R(T_{in} - T_{liq})}$$

(b) Putting in some numbers (see Table 12.1 of the text):  $V_l = 32\text{cm}^3$ ,  $T_{inv} = 51\text{K}$ ,  $\Delta H = 82\text{J}$ ,  $T_{liq} = 4.18\text{K}$ ,  $T_{in} = T_{out} = 15\text{K}$ ,  $p_{out} = 1\text{atm}$ , and:

$$\lambda = 2.5 \times 10^{-2} \frac{\text{N/m}^2}{\text{atm}} (p_{in} - 1\text{atm})$$

For  $p_{out} = 20\text{atm}$ , this implies  $\lambda = 0.48$ , a bit higher than the experimental curves shown on page 340 of the text.

**Problem 2 – Kittel 12-4** The situation is showed in the accompanying cartoon. The liquid helium is vaporized and then heated up to room temperature, before being expelled to the atmosphere by the pump. The heating is presumably accomplished by a heat exchanger so that the pump needs to provide only the vaporization enthalpy. One assumption of the problem is that the pressure right before the pump is the same as that just above the liquid, which is the vapor pressure at the temperature  $T_1$ .

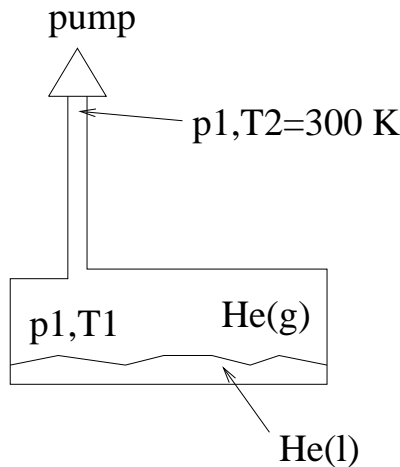


Figure 2: Cartoon of Problem 2.

The work that the pump must provide is related to the number of moles expelled by the pump per time. If the cooling load is  $0.1\text{W}$  and the pump speed is  $S = 10^{-3}\text{liters/s}$ :

$$W = (\Delta H) \frac{dn}{dt} = (\Delta H) \frac{p_1}{RT_2} \frac{dV}{dt} = (\Delta H) \frac{p_1 S}{RT_2}$$

$$\Rightarrow p_1 = \frac{WRT_2}{(\Delta H)S} = \frac{(8.314)(300)(10^{-1})(10^{-5})}{(10^{-1})(82)} \text{atm} = 3 \times 10^{-4} \text{atm} = 0.23 \text{torr}$$

Figure 10.4 of the text is a vapor pressure versus temperature curve for liquid helium. This figure implies that for  $p_1 = 0.23 \text{torr}$ ,  $T_1 \approx 1.1 \text{K}$ .

If the cooling load is  $10^{-3} \text{W}$  and the pump is faster ( $S=1 \text{ liter/s}$ ), then:

$$p_1 = \frac{(8.314)(300)(10^{-3})(10^{-5})}{(82)} \text{atm} = 3 \times 10^{-7} \text{atm} = 0.00023 \text{torr}$$

Extrapolating from figure 10.4, this corresponds to a temperature less than  $0.5 \text{K}$ , which is an improvement.

**Problem 3 – Kittel 13-1** We assume we are in the classical regime, in which case Kittel formula 13-27 holds for the concentrations:

$$\begin{aligned} n_e &= \frac{1}{2} \{ [(\Delta n)^2 + 4n_i^2]^{1/2} + \Delta n \} \\ &= \frac{1}{2} (2n_i) \left\{ \left[ 1 + \frac{(\Delta n)^2}{4n_i^2} \right]^{1/2} + \frac{\Delta n}{2n_i} \right\} \\ &\approx n_i + \frac{\Delta n}{2} \end{aligned}$$

Similarly,

$$\begin{aligned} n_h &= \frac{1}{2} \{ [(\Delta n)^2 + 4n_i^2]^{1/2} - \Delta n \} \\ &\approx n_i - \frac{\Delta n}{2} \end{aligned}$$

**Problem 4 – Kittel 13-2 (a)** We assume we are in the classical regime, in which case the law of mass action holds:  $n_e n_h = n_i^2$  (see page 26-6 of the lecture notes). Therefore,

$$\sigma = e(n_e \mu_e + n_h \mu_h) = e \left( n_e \mu_e + \frac{n_i^2}{n_e} \mu_h \right)$$

We find the minimum in the usual way:

$$\frac{d\sigma}{dn_e} = e \left( \mu_e - \frac{n_i^2}{n_e^2} \mu_h \right) = 0 \rightarrow n_e^2 = n_i^2 \frac{\mu_h}{\mu_e}$$

Plugging this in,

$$\begin{aligned} \sigma_{min} &= e \left( n_i \left( \frac{\mu_h}{\mu_e} \right)^{1/2} \mu_e + n_i \left( \frac{\mu_e}{\mu_h} \right)^{1/2} \mu_h \right) \\ &= e (2n_i \sqrt{\mu_e \mu_h}) \end{aligned}$$

**(b)** For an intrinsic semiconductor, we have that  $n_e = n_h = n_i$  so that the conductivity becomes:

$$\sigma_{int} = e n_i (\mu_e + \mu_h)$$

Therefore,

$$r = \frac{\sigma_{min}}{\sigma_{int}} = \frac{2\sqrt{\mu_e\mu_h}}{\mu_e + \mu_h}$$

(c) Plugging in the numbers, we get for Si that  $r \approx 0.88$  while for InSb,  $r \approx 0.20$ .

**Problem 5 – Kittel 13-3 (a)** In an n-type material, we have that  $n_e \gg n_h$  so that to a good approximation,  $\Delta n = n_e - n_h \approx n_e$ . Then:

$$\rho = \frac{1}{\sigma} \approx \frac{1}{en_e\mu_e} \Rightarrow \Delta n \approx n_e \approx \frac{1}{(1.6 \times 10^{-19})(20)(3900)} \approx 8 \times 10^{13} \text{cm}^{-3}$$

(b) In a p-type material, we have that  $n_e \ll n_h$  so that  $\Delta n \approx -n_h$ . Then:

$$\rho = \frac{1}{\sigma} \approx \frac{1}{en_h\mu_h} \Rightarrow \Delta n \approx -n_h \approx -\frac{1}{(1.6 \times 10^{-19})(20)(1900)} \approx -1.6 \times 10^{14} \text{cm}^{-3}$$

**Problem 6 – Kittel 13-6**

$$\begin{aligned} \frac{n_d^+}{n_d} &= \frac{1}{1 + 2 \exp((\mu - \epsilon_d)/\tau)} \\ &= \frac{1}{1 + 2 \exp((\mu - \epsilon_c + \Delta\epsilon_d)/\tau)} \\ &= \frac{1}{1 + 2 \frac{n_c}{n_d} \exp(\Delta\epsilon_d/\tau)} \\ &\approx \frac{1}{2 \frac{n_c}{n_d} \exp(\Delta\epsilon_d/\tau)} \end{aligned}$$

where we assume the ionization energy is large enough to justify the last step. We assume the most electrons come from ionized donors, i.e.  $n_e \approx n_d^+$ . Then:

$$\left(\frac{n_d^+}{n_d}\right)^2 \approx \frac{n_c}{2n_d} \exp(-\Delta\epsilon_d/\tau)$$

Therefore,

$$\frac{n_d^+}{n_d} \approx \sqrt{\frac{n_c}{2n_d}} \exp(-\Delta\epsilon_d/2\tau)$$

**Problem 7 – Kittel 13-7** An exponential doping profile means that  $n_a(x) = n_0 e^{-kx}$ . This profile is something that is controllable experimentally. At a finite temperature, some fraction of these acceptors will be ionized. If we assume the acceptors are separated enough that each acceptor is independent, then the ionized acceptors concentration should follow the same profile:  $n_a^-(x) = n_0^- e^{-kx}$ . The constants are determined by the boundary conditions:

$$\begin{aligned} n_a^-(x_1) &= n_0^- e^{-kx_1} = n_1 \\ n_a^-(x_2) &= n_0^- e^{-kx_2} = n_2 \\ &\Rightarrow n_0^- = n_1 e^{kx_1} \\ &\Rightarrow k = \frac{1}{x_2 - x_1} \ln\left(\frac{n_1}{n_2}\right) \end{aligned}$$

The ionized acceptors release holes into the system. In the absence of any external field, the distribution of holes should be the same as that for the acceptors. In other words, the net charge density should be zero so by Poisson's equation, we have a linear potential (constant electric field). If we define the zero of our potential as the value at  $x = x_1$  then, our potential may be represented as  $\phi(x) = E(x - x_1)$ . The energy of a charge in such a potential is  $e\phi(x)$ . This implies that the hole density itself should have the distribution:  $n_h(x) = n_1 e^{E(x-x_1)/\tau}$ . Comparing with the previous expression, we get:

$$E = \frac{\tau k}{e} = \frac{(300)(1.38 \times 10^{-23} \ln(10^3))}{(1.6 \times 10^{-19})(10^{-5})} \approx 1.8 \times 10^4 \text{V/cm}$$