

Homework 6 - Solutions

18th November 2004

Problem 1

The momenta of particles are as usual

$$\vec{p} = \hbar(2\pi/L)\vec{n}$$

From this

$$d^3p = \frac{(\hbar 2\pi)^{1/3}}{V} d^3n$$

Number of states in d^3n is d^3n so the number of states in d^3p is $d^3n = \frac{V}{(\hbar 2\pi)^{1/3}} d^3p = \rho d^3p$. This needs to be multiplied by two if the particles have spin 1/2. Number of fermions of each type in one state is

$$f(p) = \frac{1}{e^{(pc-\mu)/\tau} + 1}$$

At zero temperature this is a step function, it equals 1 for $p < p_F$ (or $pc < \epsilon_F$) and zero otherwise, because at zero temperature the lowest possible states are occupied.

We have

$$N = \int d^3p \rho f(p) = \rho \int^{p_F} d^3p = \frac{2V}{(\hbar 2\pi)^3} \frac{4}{3} \pi p_F^3$$

This gives

$$\epsilon_F = cp_F = \pi \hbar c \left(\frac{3n}{\pi} \right)^{\frac{1}{3}}$$

The energy is

$$U = \int d^3p \rho f(p) pc = \rho c \int^{p_F} d^3p p = \frac{Vc}{\hbar^3 \pi^2 4} p_F^4 = \frac{3}{4} N \epsilon_F$$

Now the neutron star. The notes 17-6 say that the fermi energy of electrons is about 87MeV and considering that their mass is around 0.5MeV, they are relativistic. The fermi energy of neutrons is 86MeV and of protons $87MeV \times m_e/m_p \ll 87MeV$ and considering that their mass is around 940MeV they are nonrelativistic. Thus, from notes 17-5 and above, their fermi energies are

$$\epsilon_{F,n} = \epsilon_{F,0} \left(\frac{n}{n_0} \right)^{2/3} (1-x)^{2/3}$$

$$\epsilon_{F,p} = \epsilon_{F,0} \left(\frac{n}{n_0} \right)^{2/3} \frac{m_n}{m_p} x^{2/3}$$

$$\epsilon_{F,e} = \pi \hbar c \left(\frac{3n}{\pi} \right)^{1/3} x^{1/3}$$

From the energy conservation we obtain, similarly to the notes

$$\epsilon_{F,0} \left(\frac{n}{n_0} \right)^{2/3} (1-x)^{2/3} = \epsilon_{F,0} \left(\frac{n}{n_0} \right)^{2/3} \frac{m_n}{m_p} x^{2/3} + \pi \hbar c \left(\frac{3n}{\pi} \right)^{1/3} x^{1/3} - E$$

Rewriting we get

$$\frac{m_n}{m_p} x^{2/3} = K x^{1/3} = (1-x)^{2/3} - \frac{E}{\epsilon_{F,0}} \left(\frac{n_0}{n} \right)^{2/3}$$

where

$$K = \left(\frac{n_0}{n} \right)^{1/3} \pi^{2/3} 3^{1/3} \frac{c \hbar n_0^{1/3}}{\epsilon_{F,0}} \approx 4.7$$

Now we assume x is small, solve the resulting equation and see that it is indeed small. Thus assuing it is small and because $E \ll \epsilon_{F,0}$ and $n_0 \approx n$, we can replace the right side by 1. In the left side we can write $m_n \approx m_p$. Thus we get

$$x^{2/3} + K x^{1/3} - 1 = 0$$

Solving the quadratic equation, and taking the positive solution we get

$$x = \left(\sqrt{1 + (K/2)^2} - K/2 \right)^3 \approx 0.0085$$

Problem 2 (a) We are given that liquid ${}^3\text{He}$ has density $\rho = 0.081 \text{ g/cm}^3$ and the atoms obey Fermi statistics. We compute the concentration:

$$n = (0.081) \times \frac{6 \times 10^{23}}{3} \text{cm}^{-3} = 1.62 \times 10^{22} \text{cm}^{-3},$$

the Fermi energy:

$$\begin{aligned}\epsilon_F &= \frac{\hbar^2}{2m}(3\pi^2 n)^{2/3} = \frac{(1.05 \times 10^{-34})^2}{(2)(3)(1.67 \times 10^{-27})} \left(3\pi^2 \cdot (1.62 \times 10^{28})\right)^{2/3} \text{ J} \\ &= 6.74 \times 10^{-23} \text{ J} \\ &= 4.2 \times 10^{-4} \text{ eV},\end{aligned}$$

the Fermi velocity:

$$v_F = \left(\frac{2\epsilon_F}{m}\right)^{1/2} = \left(\frac{2 \cdot (6.74 \times 10^{-23})}{3 \cdot (1.67 \times 10^{-27})}\right)^{1/2} \text{ m/s} = 164 \text{ m/s},$$

and the Fermi temperature:

$$T_F = \frac{\epsilon_F}{k_B} = \frac{6.74 \times 10^{-23}}{1.38 \times 10^{-23}} \text{ K} = 4.9 \text{ K}$$

(b) The heat capacity is given by:

$$C_V = \frac{\pi^2}{2T_F}(Nk_B T) \approx 1.006(Nk_B T)$$

where T is the temperature in Kelvin. This value is roughly a factor of 3 lower than the experimental value, which indicates something else is going on.

Problem 3 (a) The self-energy should depend only on M , R and G and should be negative because the force is attractive. The dimensional analysis gives

$$U \sim -\frac{GM^2}{R}$$

(b) In the white dwarf, the electrons (mass m) form a degenerate fermion gas. The star is neutral and so the number of protons (mass M_H) is the same as the number of electrons, and so the mass of the star is roughly $M \sim M_H N$. We get

$$K \sim N\epsilon_F \sim N \frac{\hbar^2}{2m}(3\pi^2 n)^{2/3} \sim \frac{N\hbar^2}{m} \left(\frac{N}{R^3}\right)^{2/3} \sim \frac{\hbar^2}{mR^2} N^{5/3} \sim \frac{\hbar^2 M^{5/3}}{mM_H^{5/3} R^2}$$

(c) Kittel argues that the kinetic energy of the electrons (primarily due to electrons) and the gravitational energy of the ions should be of similar orders

of magnitude. This implies:

$$\begin{aligned}\frac{GM^2}{R} &\sim \frac{\hbar^2 M^{5/3}}{mR^2 M_H^{5/3}} \\ M^{1/3}R &\sim \frac{\hbar^2}{mM_H^{5/3}G} = \frac{(1.05 \times 10^{34})^2}{(10^{-28})(1.67 \times 10^{-24})^{5/3}(6.6 \times 10^{-8})} \text{g}^{1/3}\text{cm} \\ &= 7 \times 10^{20} \text{g}^{1/3}\text{cm}\end{aligned}$$

(d) If the mass of the white dwarf is that of the sun, then part (c) implies:

$$\begin{aligned}R &\sim \frac{10^{20}}{(2 \times 10^{33})^{1/3}} \text{cm} \sim 8 \times 10^8 \text{cm} \\ \rho &\sim \frac{M}{R^3} \sim \frac{2 \times 10^{33}}{(8 \times 10^8)^3} = 4 \times 10^6 \text{g/cm}^3\end{aligned}$$

(e) If the Fermi gas is composed of neutrons instead of electrons, then the results are changed by the mass ratio:

$$(M^{1/3}R)_n \sim \left(\frac{m_e}{m_n}\right)(M^{1/3}R)_e \sim (10^{-3})(7 \times 10^{20}) \text{g}^{1/3}\text{cm} \sim 10^{18} \text{g}^{1/3}\text{cm}$$

If the neutron star had the mass of the sun, then:

$$R \sim \frac{10^{17}}{(2 \times 10^{33})^{1/3}} \times \left(\frac{10^{-3}}{10^2}\right) \text{km} \sim 8 \text{km}$$

Problem 4

The momenta of particles are as usual

$$\vec{p} = \hbar(2\pi/L)\vec{n}$$

From this

$$d^3p = \frac{(\hbar 2\pi)^{1/3}}{V} d^3n$$

Number of states in d^3n is d^3n so the number of states in d^3p is $d^3n = \frac{V}{(\hbar 2\pi)^{1/3}} d^3p = \rho d^3p$. Number of bosons in each state is

$$f(p) = \frac{1}{e^{(p^2/2m-\mu)/\tau} - 1}$$

When there is a condensate, then there is a macroscopic number N_0 of bosons in the ground state and the chemical potential is close to zero. Thus

$$\begin{aligned}
N &= N_0 + \int d^3p \rho f(p) \\
&= N_0 + \frac{V}{(\hbar 2\pi)^3} \int d^3p \frac{1}{e^{p^2/2m\tau} - 1} \\
&= N_0 + \frac{V}{\hbar^3 2\pi^2} \int_0^\infty dp \frac{p^2}{e^{p^2/2m\tau} - 1} \\
&= N_0 + \frac{V}{\hbar^3 2\pi^2} (2m\tau)^{3/2} \int_0^\infty dx \frac{x^2}{e^{x^2} - 1}
\end{aligned}$$

The total energy is

$$\begin{aligned}
U &= \frac{V}{\hbar^3 2\pi^2} \int_0^\infty dp \frac{p^2}{e^{p^2/2m\tau} - 1} \frac{p^2}{2m} \\
&= \frac{V}{\hbar^3 2\pi^2} (2m)^{3/2} \tau^{5/2} \int_0^\infty dx \frac{x^4}{e^{x^2} - 1}
\end{aligned}$$

Heat capacity at constant volume is

$$C_V = \frac{1}{k_B} \frac{dU}{d\tau} = \frac{5}{2} \frac{U}{T}$$

For process at constant volume we have

$$dU = \tau d\sigma$$

Thus

$$\sigma = \int^\tau \frac{dU}{\tau} = \frac{5}{3} \frac{V}{\hbar^3 2\pi^2} (2m)^{3/2} \tau^{3/2} \int_0^\infty dx \frac{x^4}{e^{x^2} - 1} = \frac{5}{3} \frac{U}{\tau}$$

If $\tau > \tau_e$ then we can write

$$N = \int d^3p \rho f(p) = \frac{V}{(\hbar 2\pi)^3} \int d^3p \frac{1}{e^{(p^2/2m - \mu)/\tau} - 1}$$

From this we have μ as a function of N . The energy is

$$U = \int d^3p \rho f(p) \frac{p^2}{2m} = \frac{V}{(\hbar 2\pi)^3} \int d^3p \frac{1}{e^{(p^2/2m - \mu)/\tau} - 1} \frac{p^2}{2m}$$

Then $C_V = \frac{1}{k_B} \frac{dU}{dT}$ at constant volume. $\sigma = \int^\tau k_B C_V \frac{d\tau}{\tau}$ at constant volume.

Problem 5

$$\begin{aligned}
 \langle (\Delta N)^2 \rangle &= \tau \frac{\partial \langle N \rangle}{\partial \mu} \\
 &= \tau \frac{\partial}{\partial \mu} \frac{1}{e^{(\epsilon-\mu)/\tau} - 1} \\
 &= \frac{e^{(\epsilon-\mu)/\tau}}{(e^{(\epsilon-\mu)/\tau} - 1)^2} \\
 &= \frac{e^{(\epsilon-\mu)/\tau} - 1 + 1}{(e^{(\epsilon-\mu)/\tau} - 1)^2} \\
 &= \langle N \rangle (1 + \langle N \rangle)
 \end{aligned}$$

Problem 6 We are consider a gas of 2000 atoms of ^{87}Rb . The observed Bose-Einstein transition occurs at the temperature $T_E = 170 \times 10^{-9}\text{K}$. In order to get the concentration, we use the formula for T_E given on page 18-4 of the lecture notes:

$$\begin{aligned}
 T_E &= \frac{2\pi\hbar^2}{Mk_B} \left(\frac{N}{2.612V} \right)^{2/3} \\
 \frac{N}{V} &= 2.612 \left(\frac{M}{2\pi\hbar^2} k_B T_E \right)^{3/2} \\
 &= 2.612 \left(\frac{87 \times 1.67 \times 10^{-27} \cdot 1.38 \times 10^{-23} \cdot 170 \times 10^{-9}}{2\pi(1.05 \times 10^{-34})^2} \right)^{3/2} \text{ m}^{-3} \\
 &= 2.85 \times 10^{19} \text{ m}^{-3} \\
 &= 2.85 \times 10^{13} \text{ cm}^{-3}
 \end{aligned}$$

To obtain the density, we multiply this by the mass per atom:

$$\rho = \frac{NM}{V} = (2.85 \times 10^{13})(87 \times 1.67 \times 10^{-24}) \frac{\text{g}}{\text{cm}^3} = 4.14 \times 10^{-9} \frac{\text{g}}{\text{cm}^3}$$

Assuming a spherical sample, then number of particles is given by $N = (\frac{4}{3}\pi R^3)(N/V)$ so that:

$$R = \left(\frac{3N}{4\pi(\frac{N}{V})} \right)^{1/3} = \left(\frac{3 \cdot 2000}{4\pi(2.85 \times 10^{13})} \right)^{1/3} \text{ cm} = 2.56 \times 10^{-4} \text{ cm}$$