Homework 3 - Solutions

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Framework

Average number of photons in one oscillator: $<n>=\frac{1}{e^{\hbar\omega/\tau}-1}$
Number of oscillators with frequency in the range $d^3\omega$ around $\vec{\omega}$: $\frac{V}{4\pi^3c^3}d^3\omega$
Number of oscillators with magnitude of frequency in the range $d\omega$ around ω: $\frac{V}{\pi^2c^3}\omega^2 d\omega$

Problem 1

(a) The frequency density of photons is mentioned above. However the problem can be solved even without using density's explicit form. This is done in this solution.

The photons are parametrized by their momenta $\vec{p}$. For example in the cubic box of size L they are given by $\vec{p}=(\hbar\pi/L)(n_x,n_y,n_z)$ where the n_i are positive integers. Let the number of photons per unit volume having momentum in the range $(p_x,p_x+dp_x)\times(p_y,p_y+dp_y)\times(p_z,p_z+dp_z)$ be $f(p)d^3p$ (as we know, f depends only on the magnitude of momentum, not the direction).

First let's calculate u, the termal energy per unit volume. Energy of photon with momentum $\vec{p}$ is $E=\hbar\omega=pc$. The energy of all the photons with momentum $\vec{p}$ in the range d^3p is then $pcf(p)d^3p$. The total energy is then the sum over all momenta:

$$u=\int d^3pf(p)pc=\int_0^\infty dp\int_0^\pi d\theta\int_0^{2\pi}d\phi p^2\sin(\theta)f(p)pc=4\pi c\int_0^\infty dpp^3f(p)$$

Now let's calculate the pressure P. Pressure is flux of momentum. That is, take some area A and some time interval dt. Then sum all the momenta of photons crossing this area in this time interval. Then divide by Adt. (The result obviously doesn't depend on A and dt - that's why we needed to divide by them)

To do this consider first the photons with momentum $\vec{p}$ (in the range d^3p). Suppose they enter an area A under some angle $\cos(\theta)$. Then the volume from which they enter is $Acdt\cos(\theta)$. The number of such photons is then $Acdt\cos(\theta)f(p)d^3p$. Each photon delivers momentum $p_x = p\cos(\theta)$. Thus they deliver momentum $Acdtf(p)\cos^2(\theta)d^3p$. So the total flux is sum over all momenta:

$$
\begin{aligned}
P = J &= \int d^3pf(p)pc\cos^2(\theta) \\
&= \int_0^\infty dp \int_0^\pi d\theta \int_0^{2\pi} d\phi p^2 \sin(\theta)\cos^2(\theta)f(p)pc \\
&= 2\pi c \int_0^\pi d\theta \sin(\theta)\cos^2(\theta) \int_0^\infty dpp^3 f(p) \\
&= 2\pi c \frac{2}{3} \int_0^\infty dpp^3 f(p)
\end{aligned}
$$

Comapring this with the formula for u we see $P = u/3$.

(b) The formula in the notes expresses U in terms of V and τ. To use formula $p = -(\partial U/\partial V)_\sigma$ we need to express U in terms of V and σ. Since $\sigma \sim V\tau^3$, this is done by writting

$$
U = \frac{\pi^2}{15\hbar^3c^3}V\tau^4 = \frac{\pi^2}{15\hbar^3c^3}(V\tau^3)^{4/3}V^{-1/3}
$$

Now we can differentiate U keeping $V\tau^3 \sim \sigma$ constant. Doing this we get $P = U/3V = u/3$.

To complete, let's establish the desired relations. $V = L^3 = (n\pi/\omega)^3$ gives

$$
dV/V = dL^3/L^3 = 3L^2dL/L^3 = 3dL/L = 3d(n\pi/\omega)/(n\pi/\omega) = -3d\omega/\omega
$$

As mentioned adiabatic process doesn't change the occupation number of the energy levels. That is the probablilities $p = exp(-\hbar\omega/\tau)$ don't change. So ω/τ don't change so $0 = d(\omega/\tau) = d\omega/\tau - \omega d\tau/\tau^2$ and hence $d\omega/\omega = d\tau/\tau$.

Problem 2

From "Framework" we have: Number of photons with magnitude of frequency in the range $d\omega$ around ω: $\frac{1}{e^{\hbar\omega/\tau}-1}\frac{V}{\pi^2c^3}\omega^2 d\omega$

The total number of photons is the sum of all such so

$$N = \int_0^\infty \frac{1}{e^{\hbar\omega/\tau}-1}\frac{V\omega^2}{\pi^2c^3}d\omega = \frac{V\tau^3}{\pi^2c^3\hbar^3}\int_0^\infty \frac{x^2}{e^x-1}dx \approx 2.404\frac{V\tau^3}{\pi^2c^3\hbar^3}$$

Comparing with the formula for the entropy from page 9-5 we have

$$\frac{\sigma}{N} \approx \frac{4\pi^4}{2.404\times 45} \approx 3.6$$

Problem 3

This is very similar to the problem 1a). Here the difference is that instead of momentum flux we are calculating energy flux. Thus instead of summing $p_x = p\cos(\theta)$ of each particle we are summing $E = \hbar\omega$. And we consider only particles going to the right say (e.i. out of the hole).

From "Framework" : Number of photons with frequency in the range $d^3\omega$ around $\vec{\omega}$: $\frac{1}{e^{\hbar\omega/\tau}-1}\frac{V}{4\pi^3c^3}d^3\omega$

In time dt through the area A the photons in the freqency range $d^3\omega = d\omega d\theta d\phi\omega^2\sin(\theta)$ transfer energy

$$Acdt\cos(\theta)\times\frac{1}{4\pi^3c^3}\frac{1}{e^{\hbar\omega/\tau}-1}d\omega d\theta d\phi\omega^2\sin(\theta)\times\hbar\omega$$

Thus the flux density per A, dt, $d\omega$, $d\Omega = d\theta d\phi\sin(\theta)$ is

$$j = \frac{1}{4\pi^3c^2}\frac{1}{e^{\hbar\omega/\tau}-1}\hbar\omega\cos(\theta)$$

Total radiation emitted is then

$$\begin{aligned}
J &= \int_0^\infty d\omega\int_0^{\pi/2}d\theta\int_0^{2\pi}d\phi\cos(\theta)\sin(\theta)\frac{\hbar}{4\pi^3c^2}\frac{\omega^3}{e^{\hbar\omega/\tau}-1} \\
&= 2\pi\frac{1}{2}\frac{\hbar}{4\pi^3c^2}\frac{\tau^4}{\hbar^4}\int_0^\infty dx\frac{x^3}{e^x-1} \\
&= \frac{\pi^2\tau^4}{60c^2\hbar^3} \\
&= uc/4
\end{aligned}$$

Problem 4

(a) From dimensional analysis $U \sim -\frac{Gm^2}{r} \approx -3.78\times 10^{48}$ergs.

(b) Energy of thermodynamic system is NkT and we need to equate this to $-V/2$. This gives

$$T \sim -V/2Nk \approx 1.4 \times 10^7 \text{degrees K}$$

Problem 5

(a) Single oscilator energies are $E = n\hbar\omega$ so its partition function is

$$Z = \sum e^{-n\hbar\omega/\tau} = \sum (e^{-\hbar\omega/\tau})^n = \frac{1}{e^{\hbar\omega/\tau} - 1}$$

The harmonic oscillators in the box are independent of each other and distinguishable (they have different $\vec{\omega}$) so the partition function is

$$Z = \prod Z_{\vec{\omega}} = \prod \frac{1}{e^{\hbar\omega/\tau} - 1}$$

(b)

The frequencies are $\omega = (c\pi/L)\sqrt{n_x^2 + n_y^2 + n_z^2} = cn\pi/L$.

$$\begin{aligned}
F &= -\tau \log Z \\
&= 2\tau \sum_{\vec{n}} \log(e^{\hbar\omega/\tau} - 1) \\
&\approx 2\tau \int_0^\infty \int_0^\infty \int_0^\infty d^3n \log(e^{nc\hbar\pi/\tau L} - 1) \\
&= 2\tau \frac{1}{8} 4\pi \int_0^\infty dn n^2 \log(e^{nc\hbar\pi/\tau L} - 1) \\
&= \frac{V\tau^4}{\pi^2\hbar^3c^3} \int_0^\infty dx x^2 \log(e^x - 1) \\
&= \frac{V\tau^4}{\pi^2\hbar^3c^3} \Big([\frac{x^3}{3} \log(1 - e^{-x})]_0^\infty - \frac{1}{3} \int_0^\infty \frac{x^3}{e^x - 1} dx \Big) \\
&= -\frac{\pi^2 V\tau^4}{45\hbar^3c^3}
\end{aligned}$$

The factor of 2 comes from 2 polarizations.

Problem 6

The flux between the planes before inserting the middle plane is

$$J_0 = \sigma(T_u^4 - T_l^4)$$

Let the middle plane has temperature T_m. Since the system is in equlibrium, the energy that comes to that plane has to equal the energy that comes out. That is the same as saying that the flux that comes from the left side has to equal the flux that comes out to the right side. That is

$$J_1 = \sigma(T_u^4 - T_m^4) = \sigma(T_m^4 - T_l^4)$$

This gives

$$T_m = \frac{1}{2}(T_u^4 + T_l^4)$$

Pluging this into J_1 we get

$$J_1 = \frac{1}{2}\sigma(T_u^4 - T_l^4) = \frac{1}{2}J_0$$

Problem 7

From 10-4 we have for the energy

$$U = \frac{3V}{2\pi^2\hbar^3 v^3}\tau^4 \int_0^{\theta/T} \frac{x^3 dx}{e^x - 1} = \frac{9N\tau}{x_D^3}\int_0^{x_D} \frac{x^3 dx}{e^x - 1}$$

$$\begin{aligned}
\frac{x}{e^x - 1} &= \frac{x}{x + x^2/2 + x^3/6 + \cdots} \\
&= \frac{1}{1 + (x/2 + x^2/6 + \cdots)} \\
&= 1 - (x/2 + x^2/6 + \cdots) + (x/2 + x^2/6 + \cdots)^2 + \cdots \\
&= 1 - x/2 + x^2/12 + \cdots
\end{aligned}$$

Thus

$$U = \frac{9N\tau}{x_D^3}\int_0^{x_D} x^2 - x^3/2 + x^4/12 + \cdots$$

$$U = 3N\tau(1 + \frac{3}{8}x_D + \frac{1}{20}x_D^2 + \cdots)$$

Now notice $x_D \sim 1/\tau$. Therefore the first term is proportional to τ second is independent of τ and third proportional to $1/\tau$. To get the heat capacity we need to differentiate with $T = \tau/k$. Thus the second term gives zero. For the rest we get:

$$C = 3Nk(1 - \frac{1}{20}x_D^2 + \cdots) = 3Nk(1 - \frac{1}{20}(\frac{\theta}{T})^2 + \cdots)$$

For $\theta = T$, $C_v/n \approx 23.70 \frac{\text{J}}{\text{mol K}}$ which is within 0.2% of the actual value in Table 4.2 of Kittel.

Problem 8

(a) This time, there is only one polarization instead of three. The debye frequency is obtained as follows. One counts the number of modes. Then one assumes (which is only approximately true) that density of states is that of photon gas (this time for phonons) but that the states go only until ω_D so that their number is the number of modes. In our case since we have only one mode, number of modes gets reduced by three and the density of states gets also reduced by three, so the ω_D doesn't change and hence the debye temperature doesn't change. Pluging numbers we get

$$\theta = \left(\frac{6\pi^2 N}{V}\right)^{1/3} \frac{\hbar v}{k} = 19.8\text{K}$$

(b) The total energy is the sum over energies of all modes and hence is reduced by the factor of three. Thus

$$\frac{C_v}{M} = \frac{4\pi^4 N_A k}{5 m_{He} \theta^3} T^3 = 0.0209 \times T^3$$

which is quite close to the experimental value.