

Homework 2 - Solutions

19th October 2004

Problem 1

The following are useful

$$\int_0^{\infty} dx e^{-ax^2} = \frac{1}{2} \sqrt{\frac{\pi}{a}}$$

$$\int_0^{\infty} dx x e^{-ax^2} = \frac{1}{2a}$$

Differentiating with a we get

$$\int_0^{\infty} dx x^2 e^{-ax^2} = -\frac{d}{da} \int_0^{\infty} dx e^{-ax^2} = -\frac{d}{da} \frac{1}{2} \sqrt{\frac{\pi}{a}} = \frac{1}{4a} \sqrt{\frac{\pi}{a}}$$

Similarly we get

$$\int_0^{\infty} x^3 e^{-ax^2} dx = \frac{1}{2a^2}$$

$$\int_0^{\infty} x^4 e^{-ax^2} dx = \frac{3}{8a^2} \sqrt{\frac{\pi}{a}}$$

In our case $a = 2m/\tau$. The most probable velocity is when $dp/dv = 0$ and this gives

$$v_{prob} = \sqrt{\frac{1}{a}} = \sqrt{\frac{2\tau}{m}}$$

The others

$$\langle v \rangle = \int_0^{\infty} v p(v) dv = \sqrt{\frac{8\tau}{\pi m}}$$

$$\langle v^2 \rangle = \int_0^{\infty} v^2 p(v) dv = \frac{3\tau}{m}$$

$$\sqrt{\langle v^2 \rangle} = \sqrt{\frac{3\tau}{m}}$$

N_2 has molecular weight approximately $m_{N_2} = 28m_p$ where $m_p = 1.67 \times 10^{-27}$ kg. Putting in the numbers gives

$$v_{prob} = \sqrt{\frac{2\tau}{m}} = \left[\frac{2(1.38 \times 10^{-23} \frac{J}{K}(293K))}{28 \times 1.67 \times 10^{-27} \text{kg}} \right]^{1/2} = 415 \frac{m}{s}$$

$$\langle v \rangle = 468 \frac{m}{s}$$

$$v_{rms} = 507 \frac{m}{s}$$

Problem 2

Inside the molecules have the boltzman distribution of velocities. The distribution of velocities outside is given by how many of which molecules gets out through the hole in some infinitesimal time dt . For the molecules having some velocity \vec{v} there is some volume from which the molecules of this velocity can get out (in some infinitesimal time dt). For molecules that have the same direction but twice as large magnitude of velocity, this volume is twice as large. Thus we see that the number of them that gets out is proportional to $vp(v) \sim v^3 e^{-mv^2/2\tau}$. And that is the distribution of speeds (magnitudes of velocities) of molecules that get out, we just need to normalize it - that is divide this by the integral over all speeds.

$$\int_0^\infty v^3 e^{-mv^2/2\tau} = \frac{2\tau^2}{m^2}$$

Thus the distribution of the speeds outside is

$$f(v) = \frac{m^2}{2\tau^2} v^3 e^{-mv^2/2\tau}$$

Problem 3

Consider one spin. It can have energy either E or $-E$. Thus it's partition function is

$$Z_1 = e^{-E/\tau} + e^{E/\tau}$$

The spins of the whole system are independent and distinguishable (they sit on different sites). Thus the partition function of the full system is $Z = Z_1^N$. The free energy is thus

$$F = -\tau \log Z = -N\tau \log(e^{-E/\tau} + e^{E/\tau})$$

The entropy is

$$\begin{aligned} S(\tau) &= -\left(\frac{\partial F}{\partial \tau}\right)_V = -N\left(-\ln(e^{E/\tau} + e^{-E/\tau}) - \tau \frac{-\frac{E}{\tau^2}e^{E/\tau} + \frac{E}{\tau^2}e^{-E/\tau}}{e^{E/\tau} + e^{-E/\tau}}\right) \\ &= N\left(\ln(e^{E/\tau} + e^{-E/\tau}) - \frac{E}{\tau} \tanh \frac{E}{\tau}\right) \end{aligned}$$

and the energy

$$U = F + \tau S = -NE \tanh \frac{E}{\tau}$$

Problem 4

Let the configurations of the first system be parametrized by i and of the second system by j . If the systems are weakly interacting then the configurations of the combined system are parametrized by (i, j) and the energy of such configuration is $E_{ij} = E_i + E_j$. Then we have

$$Z_{12} = \sum_{ij} e^{-(E_{ij})/\tau} = \sum_i \sum_j e^{-E_i/\tau} e^{-E_j/\tau} = \sum_i e^{-E_i/\tau} \sum_j e^{-E_j/\tau} = Z_1 Z_2$$

Problem 5

(a) The configurations are parametrized by $n = 0, \dots, N$ the number of open links and the energy of such configuration is $E_n = n\epsilon$. Using formula for geometric series

$$Z = \sum_n e^{-n\epsilon/\tau} = \frac{1 - \exp(-(N+1)\epsilon/\tau)}{1 - \exp(-\epsilon/\tau)}$$

(b) For $\epsilon/\tau \gg 1$

$$\langle n \rangle = \sum_n n e^{-n\epsilon/\tau} = 0 + e^{-\epsilon/\tau} + 2e^{-2\epsilon/\tau} + \dots = e^{-\epsilon/\tau} (1 + 2e^{-\epsilon/\tau} + \dots) \approx e^{-\epsilon/\tau}$$

Problem 6

Say there are n_+ links pointing to the right and n_- to the left. Then we have

$$n_+ + n_- = N$$

$$n_+ - n_- = 2s$$

The number of ways we can get $2s$ is

$$g(N, s) = \binom{N}{n_+}$$

We can get l when $2s = l/\rho$ or $-2s = l/\rho$. Thus the total number of states is

$$g(N, s) + g(N, -s) = \frac{2N!}{(\frac{1}{2}N + s)!(\frac{1}{2}N - s)!}$$

To find the entropy, we make use of the Stirling approximation and also the fact that $|s| \ll N$

$$\begin{aligned} \sigma(l) &= \ln(g(N, s) + g(N, -s)) \\ &= \ln 2N! - \ln(\frac{1}{2}N + s)! - \ln(\frac{1}{2}N - s)! \\ &\sim N \ln N - \frac{N}{2}(1 + \frac{2s}{N}) \ln \frac{N}{2}(1 + \frac{2s}{N}) - \frac{N}{2}(1 - \frac{2s}{N}) \ln \frac{N}{2}(1 + \frac{2s}{N}) \\ &= 2g(N, 0) - \frac{N}{2}(1 + \frac{2s}{N}) \ln(1 + \frac{2s}{N}) - \frac{N}{2}(1 - \frac{2s}{N}) \ln(1 - \frac{2s}{N}) \\ &= 2g(N, 0) - \frac{N}{2}(1 + \frac{2s}{N})(\frac{2s}{N} - \frac{2s^2}{N^2}) - \frac{N}{2}(1 - \frac{2s}{N})(-\frac{2s}{N} - \frac{2s^2}{N^2}) \\ &= 2g(N, 0) - \frac{N}{2}(1 + \frac{2s}{N})(\frac{2s}{N} - \frac{2s^2}{N^2}) + \frac{N}{2}(1 - \frac{2s}{N})(\frac{2s}{N} + \frac{2s^2}{N^2}) \\ &\sim 2g(N, 0) - \frac{2s^2}{N} \\ &= 2g(N, 0) - \frac{l^2}{2N\rho^2} \end{aligned}$$

The force is then

$$f = -\tau \left(\frac{\partial \sigma}{\partial l} \right)_U = \frac{l\tau}{N\rho^2}$$

Problem 7

The energy levels of a particle in one dimensional box are

$$E_n = \frac{\hbar^2 \pi^2}{2ML^2} n^2$$

Its partition function is given by

$$Z_1 = \sum_{n=0}^{\infty} e^{-E_n/\tau} = \int_0^{\infty} dn e^{-\alpha^2 n^2} = \frac{\sqrt{\pi}}{2\alpha} = \sqrt{\frac{M\tau}{2\pi\hbar^2}} L$$

where $\alpha = \sqrt{\frac{\hbar^2 \pi^2}{2M\tau L^2}}$. The particles are independent and indistinguishable. Therefore

$$Z_N = \frac{Z_1^N}{N!}$$

We get the free energy by the usual method:

$$\begin{aligned} F &= -\tau \ln Z_N = -\tau \left(N \ln \sqrt{\frac{M\tau}{2\pi\hbar^2}} L - N \ln N + N \right) \\ &= -\tau \left(N \ln \sqrt{\frac{M\tau}{2\pi\hbar^2}} \frac{L}{N} + N \right) \end{aligned}$$

And the entropy:

$$\begin{aligned} \sigma &= -\left(\frac{\partial F}{\partial \tau}\right)_L = N \ln \sqrt{\frac{M\tau}{2\pi\hbar^2}} \frac{L}{N} + N + \frac{N}{2} \\ &= N \left(\ln \sqrt{\frac{M\tau}{2\pi\hbar^2}} \frac{L}{N} + \frac{3}{2} \right) \end{aligned}$$

Problem 8

We will use $\frac{1}{\tau} = \left(\frac{\partial \sigma}{\partial U}\right)_V$. That is we consider a change of the system at constant volume. At constant volume, the energy levels don't change, but the distribution of probabilities does. We have

$$\sigma = - \sum p_i \log p_i$$

$$d\sigma = - \sum dp_i \log p_i - \sum p_i dp_i / p_i = \sum dp_i (1 + \lambda_1 + \lambda_2 E_i) + \sum dp_i = \lambda_2 \sum p_i E_i = \lambda_2 U$$

Where we used

$$\sum dp_i = d \sum p_i = d1 = 0$$

Thus $\lambda_2 = 1/\tau$.