Homework 1 - Solutions

19th October 2004

1

Entropy is the logarithm of the number of accesible states. If there are N chairs and k students then since students are distinguishable the number of configurations they can be in is $N!/(N-k)!$ and the entropy $\sigma = \log(N!/(N-k)!)$. Taking say $N = 70$ and $k = 40$ we have $\sigma \approx 70 \log 70 - 70 - 30 \log 30 + 30 \approx 155$. (Don't forget, the log is the natural logarithm.)

2

Define a set of configurations of the system by specifying which particles are in which side of the container. Such a configuration changes when a particle crosses the (imaginary) border separating left and right. Let the average time between changes be t. There is one configuration where all particles are on the left. There are $M = 10^{N_0/3}$ other configurations. Starting with random configuration, it takes on average approximately M steps to get to the desired configuration. Thus the time it takes to get there is Mt. Thus we just need to find t and multiply it by M.

Suppose L is the size of the box. If there were only one molecule it would cross the border (between right and left) approximately in time L/v where v is it's velocity. If there are N_0 molecules it would happen N_0 times more often so the time is $t = L/(vN_0)$.

Now you can plug in the numbers, but from the values given you can clearly see that it is in the range $(10^{-100}, 10^{100})$ even in the ages of the universe. Multiplying t and M we get for the final answer $T = 10^{2\times 10^{23} \pm 100} \approx 10^{2\times 10^{23}}$.

3

$$g(U,N,V) = CU^{\alpha N}V^{\beta N}$$

By definitions

$$\frac{1}{\tau} = \left(\frac{\partial \log g}{\partial U}\right)_{NV} = \alpha N \frac{\partial \log U}{\partial U} = \frac{\alpha N}{U}$$

$$p = \tau \left(\frac{\partial \log g}{\partial V}\right)_{NU} = \frac{\beta U}{\alpha V}$$

About ideal gas we know the following. Its energy is $U = \frac{3}{2}NkT$ and it satisfies the ideal gas equation $pV = NkT$. Heat capacity at constant volume is $C_v = \left(\frac{\partial U}{\partial T}\right)_{VN} = \frac{3}{2}Nk$.

In our gas, from the equation for the temperature above we have $U = \alpha NkT$ and comparing with ideal gas energy this gives $\alpha = 3/2$. You can also get this by expressing S in terms of T, V, N and then using $C_v = T(\partial S/\partial T)_{VN}$.

From the above equation for p we have $pV = U\beta/\alpha = \alpha N\tau\beta/\alpha = NkT\beta$ which comparing to the ideal gas equation gives $\beta = 1$.

4

$$\sigma(s) \approx \sigma_0 - 2s^2/N$$

One spin has moment m and energy mB. There are $2s$ more spins up then down. So together they have moment $M = 2sm$ and energy $U = 2smB$. Substituting $s = U/2mB$ into above we get $\sigma(U)$. Differentiating with U we get $1/\tau$. From $M = 2sm$, $M/mN = 2s/N = U/mBN$.

5

$$\sigma(n,N) = \log(g(N,n)) = \log(N+n-1)! - \log(N-1)! - \log n! \approx (N+n)\log(N+n) - N\log N - n\log n$$

Substituting $n = U/\hbar\omega$

$$\sigma(U,N) \approx (N + U/\hbar\omega)\log(N + U/\hbar\omega) - N\log N - (U/\hbar\omega)\log(U/\hbar\omega)$$

$$\frac{1}{\tau} = \frac{\partial\sigma(U,N)}{\partial U} = \cdots = \frac{1}{\hbar\omega}\log\frac{N + U/\hbar\omega}{U/\hbar\omega}$$

$$U = \frac{N\hbar\omega}{\exp(\hbar\omega/\tau) - 1}$$

6

$$\begin{aligned}
p_n &= \frac{N!}{n!(N-n)!}p^n(1-p)^{N-n} && (1)\\
&= \frac{N(N-1)\cdots(N-n+1)}{n!}p^n(1-p)^{N-n} && (2)\\
&\to \frac{(pN)^n}{n!}(1-p)^N && (3)\\
&= \frac{r^n}{n!}((1-p)^{1/p})^{Np} && (4)\\
&\to \frac{r^n}{n!}e^{-r} && (5)
\end{aligned}$$

Another solution:
We will use the following fact. From taylor expansion

$$\log(1-n/N) = -n/N + (n/N)^2/2 + \cdots$$

The three dots "$\cdots$" below mean that the term goes to zero in the limit we are considering. We have

$$\begin{aligned}
\log\frac{N!}{n!(N-n)!} &= \log N! - \log(N-n)! - \log n!\\
&\approx N\log N - N - (N-n)\log(N-n) + (N-n) - \log n!\\
&= N\log N - N(1-n/N)\log N(1-n/N) - n - \log n!\\
&= N\log N - N\log N - N\log(1-n/N) + n\log N + n\log(1-n/N) - n - \log n!\\
&= n + \cdots + n\log N + \cdots - n - \log n!\\
&\approx n\log N - \log n!
\end{aligned}$$

$$\begin{aligned}
\log p^n(1-p)^{N-n} &= n\log p + (N-n)\log(1-p)\\
&\approx n\log p - Np + np\\
&\approx n\log p - r
\end{aligned}$$

Summing the two log's we have

$$\log p_n = n\log(Np) - r - \log n!$$

$$p_n = \frac{r^n}{n!}e^{-r}$$

Let's calculate the mean and the variance. From taylor expansion

$$e^r = \sum_{n=0}^{\infty} \frac{r^n}{n!}$$

Using this in $\sum p_n$ we get $\sum p_n = 1$ as we should for a probability distribution (If we didn't we would divide it by the result of this summation).

The mean is $< n >= \sum np_n$

The variance is $< (\triangle n)^2 >=< (n- < n >)^2 >=< n^2 - 2n < n > - < n >^2 >=< n^2 > - < n >^2$

And $< n^2 >= \sum n^2 p_n$.

To calculate these we do the following. In the expansion of e^r above, take $r(d/dr)$ of both sides. On the left we get re^r, on the right $\sum nr^n/n! = e^r \sum np_n = e^r < n >$. Thus $< n >= r$.

Similarly take $r(d/dr)(r(d/dr))e^r$. It gives on the left side $(r^2 + r)e^r$ and the right side $\sum n^2 r^n/n! = e^r < n^2 >$ so $< n^2 >= r^2 + r$.

Thus $< (\triangle n)^2 >=< n^2 > - < n >^2 = r^2 + r - r^2 = r$.

Fractional uncertainity is $\sqrt{< (\triangle n)^2 >}/ < n >= 1/\sqrt{r}$.