

PHYSICS DEPARTMENT, PRINCETON UNIVERSITY

**PHYSICS 301 MIDTERM EXAMINATION**

October 20, 2004, 10:00–10:50 am, Jadwin A06

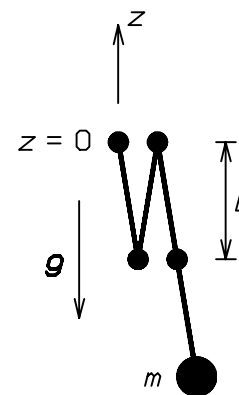
# SOLUTIONS

This exam contains two problems. Work both problems. The problems count equally although one might be harder than the other. Do all the work you want graded in the separate exam books.

Write legibly. If I can't read it, it doesn't count!

Put your name on all exam books that you hand in. (Only one should be necessary!!!) On the first exam book, rewrite and sign the honor pledge: *I pledge my honor that I have not violated the Honor Code during this examination.*

1. Molecules in a polymer are long “chains” which can tangle up or stretch out. As a very simple model, consider a chain with a large number,  $N$ , of links of length  $L$ . For convenience, take  $N$  to be an even number. One end of the chain is fastened to a fixed support and a mass  $m$  is attached to the other end of the chain. The links are massless and the whole system is in a gravitational field  $\mathbf{g}$  pointing down. In this particular chain, molecular forces allow the “hinges” between links to be completely folded or completely straight, so all the links are vertical as shown schematically in the figure. Take the gravitational energy to be zero when the mass is even with the support. Also, ignore all forms of energy other than gravitational energy and weak interaction energy with a heat bath at temperature  $\tau$ .



- (a) Without elaborate calculation, determine the average energy,  $U$ , entropy,  $\sigma$ , and free energy,  $F$ , at very low temperatures ( $\tau \rightarrow 0$ ).

————— Solution —————

In this case mechanical effects dominate thermal effects, the chain hangs straight down, and there is only one state. So the energy is a minimum,  $U = -NmgL$ ; the entropy is 0,  $\sigma = 0$ ; and  $F = -NmgL$ .

————— End Solution —————

- (b) Again without elaborate calculation, determine the energy, entropy, and free energy at very high temperatures.

————— Solution —————

In this case, thermal effects dominate mechanical effects, and all states are equally likely. The number of ways the chain can fold is  $2^N$ , so the entropy is  $\sigma = N \log 2$ . The states are symmetrically distributed with respect to the zero of energy (for every state with the mass a given distance below  $z = 0$ , there is an equivalent state with the mass the same distance above  $z = 0$ ), so the energy is  $U = 0$ . Finally, the free energy is  $F = U - \tau\sigma = -N\tau \log 2$ .

The number of ways the chain can fold which puts the mass at  $z = 0$  (the most probable energy) is  $N!/((N/2)!(N/2)!)$ . Then the entropy is (using Stirling’s approximation):

$$\sigma = N \log N - N - 2 \left( \frac{N}{2} \log \frac{N}{2} - \frac{N}{2} \right) = N \log 2.$$

So the entropy corresponding to the most probably energy is essentially the same as the entropy when any energy is allowed!

————— End Solution —————

- (c) What is the partition function for this system at (any) temperature  $\tau$ ?

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Solution

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We can rotate one link from down to up without changing any of the rest of the system. Therefore, we can treat the system as  $N$  independent links. If a link is down, its contribution to the energy is  $-mgL$ . If the link is up, its contribution to the energy is  $+mgL$ . So the partition function for one link is

$$Z_1 = e^{-mgL/\tau} + e^{+mgL/\tau} = 2 \cosh(mgL/\tau).$$

To get the partition function for  $N$  links, we just take the product of  $N$  single link partition functions. Note that there is no  $N!$  overcounting correction, since the links are distinguishable. So

$$\boxed{Z = (2 \cosh(mgL/\tau))^N}$$

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End Solution

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(d) What are the free energy, energy and entropy of this system at temperature  $\tau$ ?

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Solution

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$$F = -\tau \log Z = \boxed{-N\tau \log(2 \cosh(mgL/\tau))}.$$

$$\sigma = -\frac{\partial F}{\partial \tau} = \boxed{N \log(2 \cosh(mgL/\tau)) - N(mgL/\tau) \tanh(mgL/\tau)}.$$

$$U = F + \tau \sigma = \boxed{-NmgL \tanh(mgL/\tau)}.$$

Note that the above expressions have the same limits for small and large temperatures as deduced in parts (a) and (b).

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End Solution

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2. Sometimes electrons are confined to a plane, but otherwise free to move in the plane. Consider a system containing  $N$  electrons of mass  $m$  confined to a plane square of side  $L$  and area  $A = L \times L$ . We completely ignore the motion perpendicular to the plane and the forces confining the electrons to the plane. Also ignore interactions between electrons. Assume the electrons are in thermal contact with a bath at temperature  $\tau$ . Remember that electrons have spin  $1/2$ .

(a) Determine the density of states. That is, determine the number of states per unit energy, where the energy,  $\epsilon$ , is the kinetic energy of the two dimensional motion of the electron in the plane.

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Solution

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We can attack this question several ways. Suppose we fit standing waves in a two dimensional  $L \times L$  box. Then

$$\epsilon = \frac{\pi^2 \hbar^2}{2mL^2} (n_x^2 + n_y^2),$$

where the RHS is  $p^2/2m$ ,  $p_x = \pi\hbar n_x/L$  and  $p_y = \pi\hbar n_y/L$ .  $n_x$  and  $n_y$  are integers ( $> 0$ ) and the expressions for the momentum components correspond to fitting an integer number of half wavelengths in the box. There is one center of mass state for each pair of integers. That is the density of center of mass states in the  $n_x$ - $n_y$  plane is  $dn_x dn_y$ . There are two spin states per center of mass state, so the total density of states is  $2dn_x dn_y$ . Now change variables to  $r$  and  $\phi$ , where  $n_x = r \cos \phi$  and  $n_y = r \sin \phi$ . Then the density of states is  $2r dr d\phi$ . We can integrate over  $\phi$  (one quadrant only) to get a factor of  $\pi/2$  and the density of states is  $\pi r dr$ . Rewrite the energy in terms of  $r$  and  $\phi$ :  $\epsilon = (\pi^2\hbar^2/2mL^2)r^2$ . so  $d\epsilon = 2(\pi^2\hbar^2/2mL^2)r dr$ . Solve for  $r dr$  and plug into the density of states

$$\mathcal{D}(\epsilon) d\epsilon = \frac{mL^2}{\pi\hbar^2} d\epsilon, \quad \mathcal{D}(\epsilon) = \boxed{\frac{mA}{\pi\hbar^2}}.$$

Note that this density of states is independent of energy!

The density of states can also be found from  $d\mathcal{D} = 2dx dy dp_x dp_y / (2\pi\hbar)^2$  where the first factor of two accounts for the two spin states. Integrating over  $dx dy$  gives the area  $A$ . The momentum components can be changed to magnitude and direction (in the plane). Integrating over the direction gives  $2\pi$ . So,  $d\mathcal{D} = Ap dp / \pi\hbar^2$ . With  $d\epsilon = p dp / m$ , we get

$$\mathcal{D}(\epsilon) d\epsilon = \frac{mA}{\pi\hbar^2} d\epsilon,$$

as before.

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End Solution

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(b) Suppose the electrons are cold ( $\tau \rightarrow 0$ ). What is the chemical potential (Fermi energy)?

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Solution

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We must choose the Fermi energy to get the right number of particles. Since the electrons are cold, states below the Fermi energy are completely filled and states above the Fermi energy are completely empty. So

$$N = \int_0^{\epsilon_F} \mathcal{D}(\epsilon) d\epsilon = \frac{mA}{\pi\hbar^2} \epsilon_F.$$

So,

$$\epsilon_F = \frac{\pi\hbar^2}{m} \frac{N}{A} = \boxed{\frac{\pi\hbar^2}{m} n},$$

where  $n = N/A$  is the surface density of electrons. Note that the Fermi energy is proportional to the density rather than the  $2/3$  power of the density in the three dimensional case.

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End Solution

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(c) What is the total kinetic energy of this two dimensional gas of electrons?

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Solution

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We need to do the same integral as above, but with an extra factor of  $\epsilon$  in the integrand. This produces an extra factor of  $\epsilon_F/2$  in the result. So,

$$U_0 = \boxed{\frac{N\pi\hbar^2}{2m}n = N\frac{\epsilon_F}{2}}.$$

The average energy is half the Fermi energy rather than  $3/5$  the Fermi energy in the three dimensional case.

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End Solution

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