

Time Independent Solutions of the Diffusion Equation

In some cases, we'll be interested in the time independent solution of the diffusion equation. Why would we be interested in this? Answer: when a system reaches a steady state, the temperature distribution must satisfy

$$\nabla^2 \tau = 0,$$

as well as whatever boundary conditions exist. Suppose for example that a high (τ_1) and low (τ_2) temperature reservoir are connected by a bar with a non-uniform cross section. The transport equation applies, so $\mathbf{J}_u = -K\nabla\tau$. If we are in a steady state, then any energy entering a thin slab of the bar must leave the other side of the slab at the same rate. In other words, the energy flux (not flux density) must be constant along the length of the bar. So

$$J(x)A(x) \approx \text{constant}$$

where x measures position along the bar and $A(x)$ is the cross sectional area at position x . This says that, approximately,

$$\nabla\tau \propto \frac{1}{A(x)}.$$

As a specific example, suppose that the bar is in the form of a truncated cone, with $A(x) = A_1(x^2/x_1^2)$ where A_1 is the cross section at x_1 , and x_1 is the value of the x coordinate at the τ_1 end of the bar and $x_2 > x_1$ is the value of the x coordinate at the τ_2 end of the bar. We also suppose that $\sqrt{A_1} \ll x_1$. Basically we have a bar that is a wedge of a spherical shell of inner and outer radii x_1 and x_2 . The temperature distribution must satisfy $\nabla^2\tau = 0$ subject to the boundary conditions which are: the temperature at x_1 is τ_1 , the temperature at x_2 is τ_2 and there is no energy flux through the sides of the bar, only the ends. A little thought shows that if we actually had a complete spherical shell of inner and outer radii x_1 and x_2 at temperatures τ_1 and τ_2 , then the flux density would be radial and the solution to this problem would also solve the present bar problem. So we want the solutions for $\nabla^2\tau$ which are spherically symmetric. These are the same as the solutions for a spherically symmetric electric potential in a charge free region. We know what these are. There can be a point charge at the center of the sphere ($\Phi = 1/r$) and a constant. So the temperature as function of x must be

$$\tau(x) = \frac{C_1}{x} + C_2,$$

A little algebra finds the constants,

$$\tau(x) = \frac{x_1 x_2 (\tau_1 - \tau_2)}{x(x_2 - x_1)} + \frac{x_2 \tau_2 - x_1 \tau_1}{x_2 - x_1}.$$

Note that $\nabla\tau \propto 1/x^2$ as we wanted.

Continuity Equation for Mass

Consider a medium in which the mass density is $\rho(\mathbf{r}, t)$ and the medium moves with a bulk velocity $\mathbf{v}(\mathbf{r}, t)$. Then the mass flux density at \mathbf{r} at time t (we'll drop the explicit notation) is

$$\mathbf{J}_\rho = \rho \mathbf{v} .$$

Just like other quantities, this satisfies a continuity equation, or in this case a mass conservation equation, which says that mass flux leaving a small volume is just the rate of change of mass in the volume

$$\nabla \cdot \mathbf{J}_\rho + \frac{\partial \rho}{\partial t} = \nabla \cdot (\rho \mathbf{v}) + \frac{\partial \rho}{\partial t} = 0 .$$

Sound Waves in a Gas

In this section we'll work out the wave equation for sound waves in an ideal gas. To start with, let's consider a plane wave in the pressure. The change in pressure from its equilibrium value is

$$\delta p = \delta p_0 e^{i(\mathbf{k} \cdot \mathbf{r} - \omega t)} ,$$

where δp_0 is the amplitude of the wave. The wave is assumed to be small enough that only first order quantities need to be considered. Of, course, this means that there is a sinusoidally varying pressure gradient which accelerates the gas. Suppose there is a pressure gradient in the x direction and consider a small volume of gas in a box with cross sectional area A and length dx . Then the net force on the gas in the box is $-(\nabla p)_x A dx$. The momentum of the gas in the box at time t and position \mathbf{r} is $\rho(t, \mathbf{r}) \mathbf{v}(t, \mathbf{r}) A dx$. The rate of change of this momentum must be the force. (Note that we are ignoring viscous forces and assuming smooth flow!) When calculating the rate of change, we must note that the box is moving with velocity \mathbf{v} , so at time dt , $\mathbf{r} \rightarrow \mathbf{r} + \mathbf{v} dt$. We cancel out the $A dx$ and take the time derivative and find

$$\frac{\partial \rho v_x}{\partial t} + \mathbf{v} \cdot \nabla (\rho v_x) = -(\nabla p)_x ,$$

or generalizing to a pressure gradient in an arbitrary direction,

$$\frac{\partial \rho \mathbf{v}}{\partial t} + \mathbf{v} \cdot \nabla (\rho \mathbf{v}) = -\nabla p .$$

Now, in equilibrium, ρ is a constant, p is a constant, and $\mathbf{v} = 0$. When a wave is present, all of these have small oscillatory components. On the left hand side, we can ignore the $\mathbf{v} \cdot \nabla (\rho \mathbf{v})$ term altogether since it has two powers of velocity and is second order small. In

the other term, we can ignore the oscillatory wave in ρ since it is multiplied by the first order small velocity. To make all this explicit, let's write

$$\rho = \rho_0 + \delta\rho, \quad p = p_0 + \delta p, \quad \mathbf{v} = 0 + \delta\mathbf{v}, \quad \tau = \tau_0 + \delta\tau,$$

and so on. The quantities prefixed by δ are the small oscillatory quantities and the others are the equilibrium quantities and we will drop products of oscillatory quantities in our expressions. In particular the force equation above becomes

$$\rho_0 \frac{\partial(\delta\mathbf{v})}{\partial t} = -\nabla(\delta p).$$

If we have an ideal gas,

$$p = n\tau = \frac{\rho\tau}{m},$$

so

$$\rho_0 \frac{\partial(\delta\mathbf{v})}{\partial t} = -\frac{\tau_0}{m} \nabla(\delta\rho) - \frac{\rho_0}{m} \nabla(\delta\tau).$$

We also have

$$\delta U + p_0 \delta V = \tau_0 \delta\sigma,$$

which can be rewritten in terms of unit volume

$$\frac{\delta U}{V_0} + \frac{p_0}{V_0} \delta V = \tau_0 \frac{\delta\sigma}{V_0} = \tau_0 \delta\hat{\sigma},$$

where $\hat{\sigma}$ is the entropy per unit volume or the entropy density. Now divide by dt and remember that because we have an ideal gas, the energy density per unit volume is $\hat{C}_V\tau$,

$$\hat{C}_V \frac{\partial(\delta\tau)}{\partial t} + \frac{p_0}{V_0} \frac{\partial(\delta V)}{\partial t} = \tau_0 \frac{\partial(\delta\hat{\sigma})}{\partial t}.$$

Note also that $\rho V = \text{constant}$, so $\delta\rho/\rho_0 = -\delta V/V_0$ and

$$\hat{C}_V \frac{\partial(\delta\tau)}{\partial t} - \frac{p_0}{\rho_0} \frac{\partial(\delta\rho)}{\partial t} = \tau_0 \frac{\partial(\delta\hat{\sigma})}{\partial t}.$$

Also $\tau_0 \partial(\delta\hat{\sigma})/\partial t$ is the heat added per unit volume per unit time. This must equal the heat flow which is $K\nabla^2(\delta\tau)$. Note that earlier, in deriving the thermal diffusion equation, we ignored the $p dV$ term. This was OK, because $p dV$ work is usually quite small for a solid.

At this point, it may be useful to summarize the equations we've obtained so far. We have the continuity equation for mass, which written in terms of our small quantities, is

$$\rho_0 \nabla \cdot (\delta\mathbf{v}) = -\frac{\partial(\delta\rho)}{\partial t},$$

and the force equation

$$\rho_0 \frac{\partial(\delta \mathbf{v})}{\partial t} = -\frac{\tau_0}{m} \nabla(\delta \rho) - \frac{\rho_0}{m} \nabla(\delta \tau),$$

and the thermal diffusion equation

$$\frac{\partial(\delta \tau)}{\partial t} - \frac{p_0}{\hat{C}_V \rho_0} \frac{\partial(\delta \rho)}{\partial t} = D \nabla^2(\delta \tau).$$

Note that $D = K/\hat{C}_V$ and the thermal diffusion equation is the same as the one we discussed earlier without the density term. What we're going to do is take the time derivative of the continuity equation and the divergence of the force equation. This gives a second time derivative equal to a second space derivative which is what we need for a wave equation. There is one small problem: both $\delta \rho$ and $\delta \tau$ appear in the equation. We use the heat flow equation to relate the two. Let's consider this latter relation. The time derivative produces (order of magnitude) $1/T$ where T is the period of the wave. The Laplacian produces $1/\lambda^2$, where λ is the wavelength of the wave. So the ratio of the right hand side to the left hand side is of order

$$\frac{DT}{\lambda^2} = \frac{D}{v_s \lambda},$$

where v_s is the speed of whatever wave we're dealing with—that is, the speed of sound. We know that D is roughly the speed of sound times the mean free path, so the ratio of the right hand side to the left hand side is roughly the mean free path over the wavelength of the wave. So until we get to very short wavelength waves, we can ignore the right hand side and we have

$$\frac{\partial(\delta \tau)}{\partial t} - \frac{p_0}{\hat{C}_V \rho_0} \frac{\partial(\delta \rho)}{\partial t} = 0,$$

which means

$$\delta \tau = \frac{p_0}{\hat{C}_V \rho_0} \delta \rho.$$

What we've just shown is that sound waves (with wavelengths longer than the mean free path) are isentropic. The compressed parts of the wave are hotter than the rarefied parts of the wave, but the wave goes by so fast that there's no time for heat to flow from a compression to adjacent rarefactions. The force equation becomes

$$\begin{aligned} \rho_0 \frac{\partial(\delta \mathbf{v})}{\partial t} &= -\frac{\tau_0}{m} \nabla(\delta \rho) - \frac{p_0}{\hat{C}_V m} \nabla(\delta \rho), \\ &= -\left(\frac{\tau_0}{m} + \frac{n_0 \tau_0}{\hat{C}_V m} \right) \nabla(\delta \rho), \\ &= -\frac{(\hat{C}_V + n_0) \tau_0}{\hat{C}_V m} \nabla(\delta \rho), \\ &= -\frac{\hat{C}_p \tau_0}{\hat{C}_V m} \nabla(\delta \rho), \\ &= -\frac{\gamma \tau_0}{m} \nabla(\delta \rho). \end{aligned}$$

We've used the facts that the heat capacity per unit volume at constant pressure is just $\hat{C}_V + n_0$ and the ratio C_p/C_V is denoted by γ .

Now, we take the divergence of the force equation and we have

$$\rho_0 \frac{\partial[\nabla \cdot (\delta \mathbf{v})]}{\partial t} = -\frac{\gamma \tau_0}{m} \nabla^2(\delta \rho).$$

We take the time derivative of the continuity equation

$$\rho_0 \frac{\partial[\nabla \cdot (\delta \mathbf{v})]}{\partial t} = -\frac{\partial^2(\delta \rho)}{\partial t^2}.$$

We equate the right hand sides and find the wave equation

$$\nabla^2(\delta \rho) - \frac{1}{v_s^2} \frac{\partial^2(\delta \rho)}{\partial t^2} = 0,$$

where

$$v_s = \sqrt{\frac{\gamma \tau_0}{m}},$$

is the speed of sound in the gas.

Wave Functions for a Sound Wave

If we have a harmonic wave in a gas, then all the small quantities we've been discussing look like

$$\begin{aligned} \delta \rho &= \delta \rho_0 e^{i(\mathbf{k} \cdot \mathbf{r} - \omega t)}, \\ \delta p &= \delta p_0 e^{i(\mathbf{k} \cdot \mathbf{r} - \omega t)}, \\ \delta \tau &= \delta \tau_0 e^{i(\mathbf{k} \cdot \mathbf{r} - \omega t)}, \\ \delta \mathbf{v} &= \delta \mathbf{v}_0 e^{i(\mathbf{k} \cdot \mathbf{r} - \omega t)}, \\ \delta \mathbf{r} &= \delta \mathbf{r}_0 e^{i(\mathbf{k} \cdot \mathbf{r} - \omega t)}, \end{aligned}$$

where $\delta \mathbf{r}(\mathbf{r}, t)$ is the displacement of the gas from its equilibrium position. Note we're using \mathbf{r} as a coordinate that labels where we are and $\delta \mathbf{r}$ as a dynamical variable. This is confusing, but fairly common practice. In any case, we haven't introduced $\delta \mathbf{r}$ before. It's just

$$\delta \mathbf{r} = \int \delta \mathbf{v} dt.$$

For a complex harmonic wave, the differential operators ∇ and $\partial/\partial t$ are replaced by multiplication by $i\mathbf{k}$ and $-i\omega$ respectively. The time integral operator is replaced by

multiplying by i/ω . Given one amplitude, all the others are determined by the various equations we've been dealing with. So let's express them all in terms of δp_0 . The force equation gives

$$\delta \mathbf{v}_0 = \frac{\mathbf{k}}{\omega \rho_0} \delta p_0 = \hat{\mathbf{k}} \frac{1}{v_s \rho_0} \delta p_0 = \hat{\mathbf{k}} \frac{v_s}{\gamma} \frac{\delta p_0}{p_0},$$

where $v_s = \omega/k$ and $\hat{\mathbf{k}}$ is a unit vector in the \mathbf{k} direction. We see that velocity is parallel to the direction of the wave, so we have a longitudinal wave. With this expression for $\delta \mathbf{v}_0$, we use the continuity equation to get $\delta \rho_0$,

$$\delta \rho_0 = \rho_0 \frac{\mathbf{k} \cdot \delta \mathbf{v}_0}{\omega} = \frac{1}{v_s^2} \delta p_0.$$

Using the facts that $v_s^2 = \gamma \tau_0/m$ and $p_0 = \rho_0 \tau_0/m$, we have $v_s^2 = \gamma p_0/\rho_0$ and

$$\frac{\delta \rho_0}{\rho_0} = \frac{1}{\gamma} \frac{\delta p_0}{p_0},$$

which could have been deduced from the isentropic equation for an ideal gas, $pV^\gamma = \text{Constant}$. We use the thermal diffusion equation to find $\delta \tau_0$,

$$\delta \tau_0 = \frac{p_0}{\hat{C}_V \rho_0} \delta \rho_0 = \frac{n_0 \tau_0}{\hat{C}_V \rho_0} \delta \rho_0 = \frac{\tau_0}{\hat{C}_V m} \delta \rho_0 = \frac{\tau_0}{\hat{C}_V m} \frac{1}{v_s^2} \delta p_0 = \frac{\tau_0}{\hat{C}_V m} \frac{m}{\gamma \tau_0} \delta p_0 = \frac{1}{\gamma \hat{C}_V} \delta p_0.$$

Slightly more algebra gives

$$\frac{\delta \tau_0}{\tau_0} = \frac{\gamma - 1}{\gamma} \frac{\delta p_0}{p_0},$$

which also could have been derived from the isentropic law. Finally, the displacement is found just by integrating the velocity

$$\delta \mathbf{r}_0 = \hat{\mathbf{k}} \frac{i v_s}{\gamma \omega} \frac{\delta p_0}{p_0}.$$

Note that the pressure, density, temperature and velocity are all in phase. The displacement is 90° out of phase. You may recall from Physics 103/5 that a pressure node is a displacement anti-node and vice-versa.

Heat Losses in a Wave

The equations we developed earlier included the transfer of heat from the compressed, hot parts of the wave to the rarefied, cool parts of the wave. We argued that we could ignore the heat transfer. For very short wavelength (high frequency) waves we can't ignore the heat transfer, and we need to keep the $D\nabla^2\delta\tau$ term. Rather than try to derive a wave equation, it's probably easier just to plug the plane wave solutions (as in the previous section) into the various equations. We have three equations, (continuity, force, heat transfer) and three unknowns, $\delta\mathbf{v}_0$, $\delta\tau_0$, and $\delta\rho_0$. The force equation gives

$$-i\omega\rho_0\delta\mathbf{v}_0 + i\mathbf{k}\frac{\tau_0}{m}\delta\rho_0 + i\mathbf{k}\frac{\rho_0}{m}\delta\tau_0 = 0.$$

This tells us that we have a longitudinal wave, so we'll just drop the vector symbols on \mathbf{k} and $\delta\mathbf{v}_0$ for now, The continuity equation is

$$ik\rho_0\delta v_0 - i\omega\delta\rho_0 = 0.$$

The thermal diffusion equation is

$$(k^2D - i\omega)\delta\tau_0 + i\omega\frac{p_0}{\hat{C}_V\rho_0}\delta\rho_0 = 0.$$

Now, these are homogeneous equations which have a non-trivial solution only if the determinant of the matrix of coefficients is 0. This gives the characteristic equation

$$(k^2D - i\omega)\left(\frac{k^2\tau_0}{m} - \omega^2\right) - ik^2\omega\frac{p_0}{m\hat{C}_V} = 0.$$

After a fair amount of algebra, this can be put in the form

$$\omega^2 = k^2\frac{\tau_0}{m}\frac{\hat{C}_p + ik^2K/\omega}{\hat{C}_V + ik^2K/\omega}.$$

So, the equations have a solution provided k and ω satisfy this dispersion relation. Note that when the terms containing the thermal conductivity are ignored we get the previous relation between ω and k . When they are important, we have a complex wave number and an attenuation of the wave resulting from the wave energy being dissipated as heat.

Some things we've ignored: If the gas is not monatomic, the excitation of the rotational and vibrational modes can get out of phase with the wave. In this case, energy may be lost to the excitation of these modes. This is discussed in K&K, but we don't have time.