Reading

Finish K&K chapter 7 and start on chapter 8. Also, I'm passing out several *Physics Today* articles. The first is by Graham P. Collins, August, 1995, vol. 48, no. 8, p. 17, "Gaseous Bose-Einstein Condensate Finally Observed." This describes the research leading up the first observation of a BE condensate that's not a superfluid or superconductor. The second is by Barbara Goss Levi, March, 1997, vol. 50, no. 3, p. 17, "Bose Condensates are Coherent Inside and Outside an Atom Trap," describing the first "atom laser" which was based on a BE condensate. The third is also by Levi, October, 1998, vol. 51, no. 10, p. 17, "At Long Last, a Bose-Einstein Condensate is Formed in Hydrogen," describing even more progress on BE condensates.

In addition, there is a recent *Science* report on an atomic Fermi Gas, DeMarco, B., and Jin, D. S., September 10, 1999, vol. 285, p. 1703, "Onset of Fermi Degeneracy in a Trapped Atomic Gas."

Bose condensates and Fermi degeneracy are current hot topics in Condensed Matter Research. Searching the preprint server or "Googling" with appropriate keywords is bound to turn up many more articles.

More on Fermi Gases

So far, we've considered the zero temperature Fermi gas and done an approximate treatment of the low temperature heat capacity of Fermi gases. The zero temperature Fermi gas was straightforward. We simply said that all states, starting from the lowest energy state, are filled until we run out of particles. The energy at which this happens is called the Fermi energy and is the same as the chemical potential at 0 temperature, $\epsilon_F = \mu(\tau = 0)$. Basically, all we had to do was determine the density of states, a problem we've dealt with before.

Working on the low temperature heat capacity required an approximate calculation of the energy versus temperature for a cold Fermi gas. In this calculation we assumed that the density of states near the Fermi energy is constant and this allows one to pull the density of states out of the integral and also to set the chemical potential to its 0 temperature value.

These approximations work quite well for the electron gas in metals at room temperature because the Fermi temperature for these electron is typically several tens of thousands of Kelvins.

To calculate the energy, etc., at arbitrary temperatures, one must numerically integrate the Fermi-Dirac distribution times the density of states to obtain the number of particles. Then the chemical potential is varied until the desired number of particles is obtained. Knowing the chemical potential, one can integrate the density of states times the Fermi-Dirac distribution times the energy to get the energy at a given temperature. All of this requires numerical integration or approximate techniques.

Figure 7.9 and tables 7.2 and 7.3 of K&K demonstrate that the low temperature heat capacities (low enough that the Debye lattice vibrations are accurately following a T^3 heat capacity) have a component proportional to the temperature and list the proportionality constants for various metals. One thing you will notice is that the proportionality constants agree with the calculations to only $\sim 30\%$ and up to a factor of 2 in at least one case. This is most likely due to the fact that the electrons are not really a non-interacting gas. Also, there are effects due to the crystal structure such as energy bands and gaps.

Other Fermi Gases

In addition to the conduction electron gas in metals, Fermi gases occur in other situations.

In heavy elements, the number of electrons per atom becomes large enough that a statistical treatment is a reasonable approximation. This kind of treatment is called the *Thomas-Fermi* (or sometimes the Fermi-Thomas) model of the atom.

Also in heavy elements, the number of nucleons (neutrons and protons) in the nucleus is large and, again, a statistical treatment is a reasonable approximation. The radius of a nucleus is

$$R \approx (1.3 \times 10^{-13}\ \text{cm}) \cdot A^{1/3}\ ,$$

where A is the number of nucleons. The coefficient in this relationship can vary by a tenth or so depending on just how one measures the size—scattering by charged particles, scattering by neutrons, effects on atomic structure, etc. Aside: the unit of length 10^{-13} cm which is one femtometer is called the Fermi in nuclear physics. The volume of a nucleus is

$$V = \frac{4\pi}{3} 2.2 \times 10^{-39} A\ \text{cm}^3\ ,$$

and the number density or concentration is

$$n_{\text{nuc}} = \frac{A}{V} = 1.1 \times 10^{38}\ \text{cm}^{-3}\ .$$

The nuclear density (with this number of significant digits, the mass difference between neutrons and protons is negligible) is

$$\rho_{\text{nuc}} = 1.8 \times 10^{14}\ \text{g}\,\text{cm}^{-3}\ .$$

Basically, all nuclei have the same density. Of course, this is not quite true. Nuclei have a shell structure and "full shell" nuclei are more tightly bound than partially full shell nuclei. Also, the very lightest nuclei show some deviations. Nevertheless, the density variations aren't large and it's reasonable to speak of **the** nuclear density.

The neutron to proton ratio in nuclei is about 1 : 1 for light nuclei up to about 1.5 : 1 for heavier nuclei. Assuming the latter value, then it is the neutrons whose Fermi energy is important.

$$\epsilon_F = \frac{\hbar^2}{2m_n}\left(3\pi^2(0.6\cdot n_{\rm nuc})\right)^{2/3} = 5.2\times 10^{-5}\ {\rm erg} = 32\ {\rm MeV}\,.$$

This is a little larger than K&K's number because it's computed for a nucleus with 40% protons and 60% neutrons, instead of equal numbers. Since the average kinetic energy in a Fermi gas is $3\epsilon_F/5$, the average kinetic energy is about 19 MeV in a heavy nucleus. The experimentally determined binding energy per nucleon is about 8 MeV. This varies somewhat, especially for light nuclei; it reaches a peak at ^{56}Fe. To the extent that the binding energy per nucleon and the kinetic energy per nucleon are constant, the potential energy per nucleon is also constant. This reflects the fact that the nuclear force is the short range strong force and nuclei only "see" their nearest neighbors. The strong force is about the same between neutrons and protons, between protons and protons and between neutrons and neutrons. But, the protons have a long range electromagnetic interaction. As the number of particles goes up the "anti-binding" energy of the protons goes up faster than the number of protons (can you figure out the exponent?) so the equilibrium shifts to favor neutrons in spite of the fact that they are slightly more massive than protons.

The Fermi temperature for neutrons in a heavy nucleus is

$$T_F = \epsilon_F/k = 3.8\times 10^{11}\ {\rm K}\,,$$

so nuclei (which are usually in their ground state) are very cold!

In a star like the Sun, gravity is balanced by the pressure of a very hot, but classical, ideal gas. The Sun has a mass about 300,000 times that of the Earth and a radius about 100 times that of the Earth, so the average density of the Sun is somewhat less than that of the Earth (it's about the density of water!). The temperature varies from about 20 million Kelvins at the center to about 6000 K at the surface. So it's completely gaseous and the electrons are non-degenerate throughout. Since the sun is radiating, it is cooling. Energy is supplied by nuclear reactions in the Sun's core.

A typical white dwarf star has about the mass of the Sun but the radius of the Earth. It's the degeneracy pressure of the electrons that balances gravity in a white dwarf. White dwarves shine by cooling. There are no nuclear reactions in the core, so after they cool enough, they become invisible. White dwarves are discussed in K&K, so let's move on to neutron stars which are not discussed in K&K.

Neutron Stars

In a neutron star it's the degeneracy pressure of the neutrons that balances gravity. A typical neutron star has a mass like the Sun ($M_\odot = 2 \times 10^{33}$ g) but a radius smaller than New Jersey, let's say $R \approx 10$ km. Let's assume that the mass in a neutron star is uniformly distributed. What's the density?

$$\rho = M/V = 4.8 \times 10^{14}\ \mathrm{g\,cm^{-3}}\ ,$$

about three times nuclear density. (Of course, the density in a star is not uniform and it may exceed 10 times nuclear density in the center, but we're just trying to do a back of the envelope calculation here.) In terms of the concentration of neutrons, this corresponds to

$$n_0 = 2.9 \times 10^{38}\ \mathrm{cm^{-3}}\ .$$

The Fermi energy for these neutrons is

$$\epsilon_{F,0} \approx 86\ \mathrm{MeV}\ ,$$

and the Fermi temperature is

$$T_F = 10^{12}\ \mathrm{K}\ .$$

Neutron stars are nowhere near this hot. Otherwise they would be very strong sources of gamma rays. Instead they are thought to have temperatures of millions of degrees and radiate X-rays. I believe there are some observations which indicate this. Also, due to having a magnetic field and rotating, they can radiate electromagnetic energy and are observed as pulsars. In any case, the neutrons in a neutron star are cold!

An interesting question is why is a neutron star made of neutrons? (Well, if it weren't, we probably wouldn't call it a neutron star, but besides that?) In particular, what's wrong with the following? Let the star be made of protons and electrons, each with the concentration we've just calculated. Then the star is electrically neutral because there is a sea of positively charged protons and a sea of negatively charged electrons. But the protons have a slightly lower mass than the neutrons and this is true even if one adds in the mass of the electron, so this configuration would seem to be energetically favored over the neutron configuration. In fact, a free neutron decays according to

$$n \rightarrow p + e^- + \bar{\nu}_e\ ,$$

where $\bar{\nu}_e$ is the *electron anti-neutrino*. The half life is about 15 minutes. Neutrinos are massless (or very nearly so) and for our purposes we can ignore them. That is, we can assume that the neutrons are in equilibrium with the protons and electrons. If we need to change a neutron into a proton and electron, the above reaction will do it. If we need to change a proton and electron into a neutron, there is

$$p + e^- \rightarrow n + \nu_e\ .$$

What would be the Fermi energies of the protons and electrons in our hypothetical star? The Fermi energy for the protons would be very nearly the same as that for the neutrons above (because the concentration would be the same and the mass is nearly the same). On the other hand, the Fermi energy of the electrons would be larger by the ratio of the neutron mass to the electron mass, a factor of 1838, so the electron Fermi energy would be about 160,000 MeV, enough to make about 170 nucleons! Remember that the chemical potential (the Fermi energy since all the gases are cold) is the energy required to add a particle to a system. If neutrons are in equilibrium with protons and electrons, then the chemical potential of the neutrons equals the chemical potential of the protons plus the chemical potential of the electrons minus the energy difference between a neutron and a proton plus electron. In other words

$$\epsilon_{F,n} = \epsilon_{F,p} + \epsilon_{F,e} - (m_n - m_p - m_e)c^2 \; .$$

Denote the concentrations of the neutrons, protons, and electrons by n_n, n_p, and n_e. Then

$$n_p = n_e \; ,$$

for charge neutrality and

$$n_p + n_n = n \; ,$$

where n is the concentration of the nucleons, which is not changed by the reactions above. To simplify the notation a bit, let

$$x = \frac{n_p}{n} = \frac{n_e}{n} \; , \qquad\qquad 1 - x = \frac{n_n}{n} \; .$$

Each of the Fermi energies can be written in terms of the concentrations

$$\begin{aligned}
\epsilon_{F,n} &= \frac{\hbar^2}{2m_n}(3\pi^2 n_n)^{2/3} = \epsilon_{F,0}\left(\frac{n}{n_0}\right)^{2/3}(1-x)^{2/3} \; , \\
\epsilon_{F,p} &= \frac{\hbar^2}{2m_p}(3\pi^2 n_p)^{2/3} = \epsilon_{F,0}\left(\frac{n}{n_0}\right)^{2/3}\frac{m_n}{m_p}x^{2/3} \; , \\
\epsilon_{F,e} &= \frac{\hbar^2}{2m_e}(3\pi^2 n_e)^{2/3} = \epsilon_{F,0}\left(\frac{n}{n_0}\right)^{2/3}\frac{m_n}{m_e}x^{2/3} \; ,
\end{aligned}$$

where

$$\epsilon_{F,0} = \frac{\hbar^2}{2m_n}(3\pi^2 n_0)^{2/3} \; ,$$

is the Fermi energy for a pure neutron gas at the concentration n_0 we calculated previously. We plug these energies into the energy equation to obtain

$$\epsilon_{F,0}\left(\frac{n}{n_0}\right)^{2/3}(1-x)^{2/3} = \epsilon_{F,0}\left(\frac{n}{n_0}\right)^{2/3}\left(\frac{m_n}{m_p}+\frac{m_n}{m_e}\right)x^{2/3} - E \; ,$$

where $E = (m_n - m_p - m_e)c^2 = 0.783$ MeV is the mass energy excess of a neutron over a proton and electron. If we rearrange slightly, we obtain

$$(1-x)^{2/3} = \left(\frac{m_n}{m_p} + \frac{m_n}{m_e}\right) x^{2/3} - \frac{E}{\epsilon_{F,0}} \left(\frac{n_0}{n}\right)^{2/3} ,$$

or

$$(1-x)^{2/3} = (1.0014 + 1838.7)\, x^{2/3} - \frac{0.783}{86} \left(\frac{n_0}{n}\right)^{2/3} ,$$

or

$$x^{2/3} = 0.000544 \left((1-x)^{2/3} + 0.0091 \left(\frac{n_0}{n}\right)^{2/3} \right) .$$

If n is in the neighborhood of n_0, then x is small, we can ignore x on the right hand side, and we finally obtain

$$x \approx 1.3 \times 10^{-5} .$$

At higher concentrations x will get slightly smaller and at lower concentrations x will grow slowly. The concentration of neutrons, protons, and electrons are equal ($x = 0.5$) when

$$n = 2.2 \times 10^{-8} n_0 = 6.4 \times 10^{30}\ \mathrm{cm}^{-3} .$$

Such low concentrations will be attained only very near the surface of the neutron star. Caveats: (1) The Fermi energy of the electrons works out to be about 87 MeV, so the electrons are extremely relativistic, so we really shouldn't be using our non-relativistic formula for the electron Fermi energy. One of this week's homework problems gives you a chance to modify the treatment to allow for relativistic electrons. (2) With an electron and proton instead of a neutron, the pressure changes, so the equilibrium condition that we set up is not quite right. Nevertheless, this calculation gives the flavor of what's involved and points to the correct conclusion: For most of its volume a neutron star is almost pure neutrons!

Of course, we can turn the earlier question around: how is it that nuclei have any protons??? Haven't we just shown that at nuclear densities, the nucleons must exist as neutrons, not protons???