

## Johnson Noise

This is another application of the thermal equilibrium of electromagnetic modes. Consider an ideal transmission line, like a long piece of lossless coaxial cable. Suppose its length is  $L$  and suppose it is shorted out at each end. Then any wave that travels along the line is reflected at each end and to have an appreciable amplitude, the length of the line must contain an integral number of half wavelengths. In other words this is just a one-dimensional cavity of length  $L$ . There are modes of the electromagnetic fields, most conveniently represented by the potential difference between the inner and outer conductors.  $V_n = V_{n,0} \sin(\omega t) \sin(n\pi x/L)$ , where  $n$  is any positive integer.  $V_n$  and  $V_{n,0}$  represent the potential difference and the amplitude of the potential difference. The fields must satisfy Maxwell's equations, so  $\omega = n\pi c/L$ . Actually, if the coax is filled with a dielectric, the speed of propagation can be different from  $c$ , let's assume it's filled with vacuum. If this line is in thermal equilibrium at temperature  $\tau$ , each mode acts like an oscillator and has average energy  $\hbar\omega/(e^{\hbar\omega/\tau} - 1)$ . Let's consider the low frequency limit so the average energy in each mode is just  $\tau$ . The number of modes per integer  $n$  is just 1. Then the number of modes per unit frequency is

$$n(\omega) d\omega = \frac{L}{\pi c} d\omega.$$

The energy per unit length per unit frequency is then

$$u_\omega = \frac{\tau}{\pi c},$$

at low frequencies.

As you may know, all transmission lines have a characteristic impedance,  $R$ . If a resistor  $R$  is connected across the end of the line, then a wave traveling down the line is completely absorbed by the resistor. So, let's take a resistor, in equilibrium at temperature  $\tau$ , and connect it to the end of the line. Since the resistor and the line are at the same temperature, they are already in thermal equilibrium and no net energy transfer takes place. Each mode in the line is a standing wave composed equally of traveling waves headed in both directions. The waves traveling towards the resistor will be completely absorbed by the resistor. This means that the resistor must emit waves with equal power in order that there be no net transfer of energy. The energy in the frequency band  $d\omega$  per unit length headed towards the resistor is  $\tau d\omega/2\pi c$ . This is traveling at speed  $c$ , so the power incident on the resistor is  $\tau d\omega/2\pi$  which is also the power emitted by the resistor,

What we've established so far is that the line feeds power  $\tau d\omega/2\pi$  into the resistor and vice-versa. This means that a voltage must appear across the resistor. This will be a fluctuating voltage with mean 0 since it's a random thermal voltage. However, its mean square value will not be zero. Let's see if we can calculate this. As an equivalent circuit, we have a resistor  $R$ , a voltage generator (the thermally induced voltage source),

a filter (to limit the frequencies to  $d\omega$ ), and another resistor of resistance  $R$  representing the transmission line. Then the current is  $I = V/2R$ . The average power delivered to the resistor is then  $\langle I^2 \rangle R = \langle V^2 \rangle / 4R = \tau d\omega / 2\pi$ . In the lab, one measures frequencies in Hertz rather than radians per second,  $\nu = \omega / 2\pi$ . Finally

$$\langle V^2 \rangle = 4R\tau d\nu .$$

This relates the mean square noise voltage which appears across a resistor to the temperature, resistance, and bandwidth ( $d\nu$ ). Of course, this voltage results from fluctuations in the motions of electrons inside the resistor, but we calculated it by considering electromagnetic modes in a one-dimensional cavity, a much simpler system! This thermal noise voltage is called Johnson noise.

### Debye Theory of Lattice Vibrations

A little thought will show that sound waves in a solid are not all that different from electromagnetic waves in a cavity. Further thought will show that there are some important differences that we must take into account.

The theory of lattice vibrations that we'll discuss below applies to the ion lattice in a conductor. In addition, one needs to account for the thermal effects of the conduction electrons which behave in many respects like a gas. We'll consider the electron gas later in the term. For now, we imagine that we're dealing with an insulator.

We will treat crystalline solids. This is mainly for conceptual convenience, but also because we want reasonably well defined vibrational modes.

As a model, suppose the atoms in a solid are arranged in a regular cubic lattice. Each atom vibrates around its equilibrium position. The equilibrium and the characteristics of the vibrations are determined by interactions with the neighboring atoms. We can imagine that each atom is connected to its six nearest neighbors by springs. At first sight, this seems silly. But, the equilibrium position is determined by a minimum in the potential energy, and the potential energy almost surely increases quadratically with displacement from equilibrium. This gives a linear restoring force which is exactly what happens with a spring. So our solid is a large number of coupled oscillators. In general, the motion of a system of coupled oscillators is very complex. You probably know from your classical mechanics course, that the motion of a system of coupled oscillators can be resolved into a superposition of normal modes with the motion of each mode being simple harmonic in time. So, we can describe the motion with the  $N$  vectors  $\mathbf{r}_i$  which represent the displacement of each atom from its equilibrium position, or we can describe the motion with  $3N$  normal mode amplitudes. For those of you that know about Fourier transforms, the normal modes are just the Fourier transforms of the position coordinates.

These normal modes represent elastic vibrations of our solid. They are standing elastic waves, or standing sound waves. In this respect, they are similar to the cavity modes we discussed earlier. There are two main differences. First, there are three polarizations: there are two transversely polarized waves (as we had in the electromagnetic case) and one longitudinally polarized wave (absent in the electromagnetic case). Second, there is a limit to the number of modes. If our solid contains  $N$  atoms, there are  $3N$  modes. In the electromagnetic case, there is no upper limit to the frequency of a mode. High frequency modes with  $\hbar\omega \gg \tau$  are not excited, but they are there. In the elastic case, frequencies which are high enough that the wavelength is shorter than twice the distance between atoms do not exist.

For simplicity, we are going to assume that the velocity of sound is isotropic and is the same for both transverse and longitudinal waves. Also, we'll assume that the elastic properties of the solid are independent of the amplitude of the vibrations (at least for the amplitudes we'll be dealing with).

We'll carry over as much stuff from the electromagnetic case as we can. A typical mode will look something like

$$\text{displacement component} = A \sin \omega t \sin \frac{n_x \pi x}{L} \sin \frac{n_y \pi y}{L} \sin \frac{n_z \pi z}{L},$$

where the sine factors might be cosines depending on the mode,  $A$  represents an amplitude, and for convenience, the solid is a cube of side  $L$ . The frequency and mode numbers are related by the speed of sound,  $v$ ,

$$\omega^2 = \frac{\pi^2 v^2}{L^2} (n_x^2 + n_y^2 + n_z^2).$$

If the solid contains  $N$  atoms, the distance between atoms is  $L/N^{1/3}$ . The wavelength must be longer than twice this distance. More precisely

$$\frac{2L}{n_x} > \frac{2L}{N^{1/3}},$$

with similar relations for  $n_y$  and  $n_z$ . In other words, the mode numbers  $n_x$ ,  $n_y$ , and  $n_z$  are integers within the cube  $N^{1/3} \times N^{1/3} \times N^{1/3}$ . This lower limit on the wavelength (upper limit on the frequency) is an example of the Nyquist limit discussed later in these notes.

The number of modes per unit frequency is just as it was for the electromagnetic case except that we must multiply by  $3/2$  to account for the three polarizations instead of two,

$$n(\omega) d\omega = \frac{3V\omega^2}{2\pi^2 v^3} d\omega.$$

This works for frequencies low enough that the corresponding  $n$ 's are within the cube. It's messy to deal with this cubical boundary to the mode number space. Instead, let's

approximate the upper boundary as the surface of a sphere which gives the same number of modes. In other words, there will be an upper limit to the frequency, called  $\omega_D$ , such that

$$3N = \int_0^{\omega_D} n(\omega) d\omega = \frac{3V}{2\pi^2 v^3} \int_0^{\omega_D} \omega^2 d\omega = \frac{V}{2\pi^2 v^3} \omega_D^3,$$

which gives

$$\omega_D = \left( \frac{6\pi^2 N}{V} \right)^{1/3} v.$$

Each mode acts like a harmonic oscillator and its energy is an integer times  $\hbar\omega$ . The quanta of sound are called phonons. A solid contains a thermally excited phonon gas. The average energies of these oscillators are just as they were in the electromagnetic case. We find the total energy by adding up the energies in all the modes,

$$\begin{aligned} U &= \frac{3V}{2\pi^2 v^3} \int_0^{\omega_D} \omega^2 \frac{\hbar\omega}{\exp(\hbar\omega/\tau) - 1} d\omega, \\ &= \frac{3V}{2\pi^2 \hbar^3 v^3} \tau^4 \int_0^{x_D} \frac{x^3 dx}{e^x - 1}, \end{aligned}$$

where

$$x_D = \frac{\hbar\omega_D}{\tau} = \left( \frac{6\pi^2 N}{V} \right)^{1/3} \frac{\hbar v}{\tau} = \frac{k\theta}{kT} = \frac{\theta}{T},$$

where  $\theta$  is called the Debye temperature and is given by

$$\theta = \left( \frac{6\pi^2 N}{V} \right)^{1/3} \frac{\hbar v}{k}.$$

The Debye temperature is not a temperature you can change by adding or removing heat from a solid! Instead, it's a characteristic of a given solid. The way to think of it is that a vibration with phonon energy equal to  $k\theta$  is the highest frequency vibration that can exist within the solid. Otherwise the wavelength would be too short. (The weird factor of  $6\pi^2$  occurs because we replaced a cube with a sphere!) Typical Debye temperatures are a few hundred Kelvin.

The limit of integration depends on the temperature, so in general, we can't look up the integral. Instead, we have to numerically integrate and produce a table for different values of  $x_D = \theta/T$ . Such a table is given in K&K.

There are two limiting cases where we can do the integral. The first case is very low temperature ( $T \ll \theta$ ). In this case  $x_D$  is very large and we can replace  $x_D$  by  $\infty$ . Then the integral is  $\pi^4/15$  and we have

$$U = \frac{\pi^2 V}{10\hbar^3 v^3} \tau^4 = \frac{3\pi^4 N}{5k^3 \theta^3} \tau^4.$$

The heat capacity at constant volume is then

$$C_V = \frac{12}{5} \pi^4 N k \left( \frac{T}{\theta} \right)^3 .$$

So a prediction of this theory is that at low temperatures, the heat capacities of solids should be proportional to  $T^3$ . This is borne out by experiment!

The other limit we can consider is very high temperature ( $T \gg \theta$ ). In this case, we expect all modes are excited to an average energy  $\tau$ , so the total should be  $U = 3N\tau$ . Is this what we get? At very high temperatures,  $x_D \ll 1$ , so we can expand the exponential in the denominator of the integrand,

$$\begin{aligned} U &= \frac{3V}{2\pi^2 \hbar^3 v^3} \tau^4 \int_0^{x_D} \frac{x^3 dx}{e^x - 1} , \\ &= \frac{3V}{2\pi^2 \hbar^3 v^3} \tau^4 \int_0^{x_D} x^2 dx , \\ &= \frac{V}{2\pi^2 \hbar^3 v^3} \tau^4 x_D^3 , \\ &= \frac{V}{2\pi^2 \hbar^3 v^3} \tau^4 \frac{\hbar^3 v^3}{\tau^3} \frac{6\pi^2 N}{V} , \\ &= 3N\tau , \end{aligned}$$

as expected. Actually, we picked  $\omega_D$  so this result would occur “by construction.” The heat capacity goes to  $3Nk$  in this limit.

In one of the homework problems you are asked to come up with a better approximation in the limit of small  $x_D$ .

### The Nyquist Frequency

We imagine that we have a function of time that we sample periodically, every  $T$  seconds. Then the Nyquist frequency is the frequency corresponding to a period of two samples.  $\omega_N = 2\pi/2T = \pi/T$ . Consider a sine wave at some frequency  $\omega$ ,

$$y(t) = \sin(\omega t + \phi).$$

Since we are sampling, we don't have a continuous function, but a discrete set of values:

$$y_m = \sin(\omega mT + \phi).$$

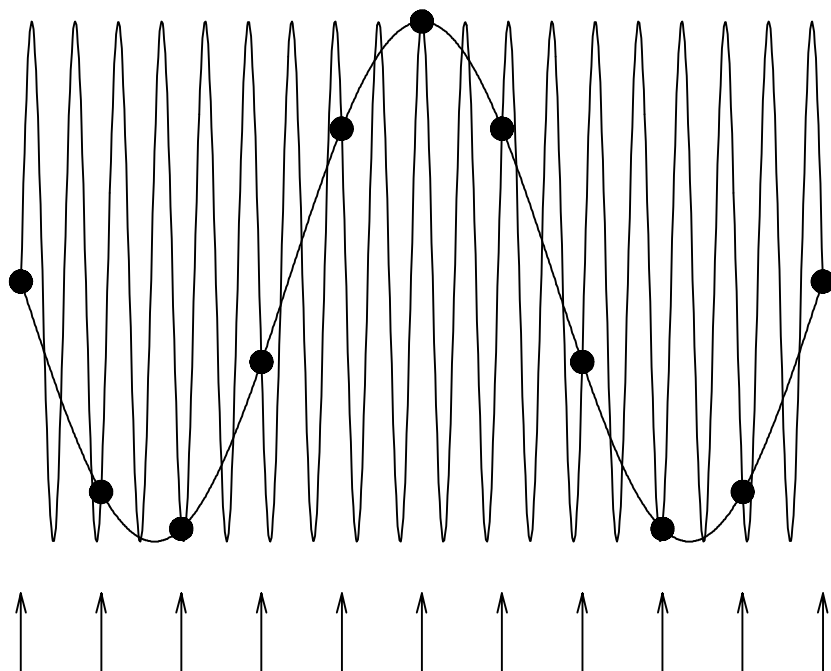
Suppose the frequency is larger than the Nyquist frequency. Then we can write it as an even integer times the Nyquist frequency plus a frequency less than the the Nyquist frequency:

$$\omega = 2n\omega_N + \Omega = 2\pi n/T + \Omega,$$

where  $-\omega_N \leq \Omega \leq +\omega_N$ . Then

$$y_m = \sin(2\pi nm + \Omega mT + \phi) = \sin(\Omega mT + \phi).$$

In other words, when we sample a sine wave periodically, waves with frequencies greater than the Nyquist frequency look exactly the same as waves with frequencies less than the Nyquist frequency. This is illustrated in the figure. The arrows along the bottom indicate



the times at which the signal is sampled. A signal at the Nyquist frequency would have one cycle every two sample intervals. The high frequency wave has 3.7 cycles every two

sample intervals or a phase change of  $3.7\pi = (4 - 0.3)\pi$  every sample. We can't tell how many multiples of  $2\pi$  go by between samples, so the high frequency wave looks exactly like a low frequency wave with  $-0.3$  cycles per two samples. The points show the value of the signal (either wave) at each sampling interval. Of course the application to the Debye theory of lattice vibrations is that the Nyquist **spatial** frequency is the highest frequency a periodic lattice can support.