PHYSICS DEPARTMENT, PRINCETON UNIVERSITY

# PHYSICS 301 FINAL EXAMINATION

January 13, 2005, 7:30–10:30pm, Jadwin A10

# SOLUTIONS

This exam contains five problems. Work any three of the five problems. All problems count equally although some are harder than others. Do all the work you want graded in the separate exam books. *Indicate clearly which three problems you have worked and want graded.* I will only grade three problems. If you hand in more than three problems without indicating which three are to be graded, I will grade the first three, only!

Write legibly. If I can't read it, it doesn't count!

Put your name on all exam books that you hand in. (Only one should be necessary!!!) On the first exam book, rewrite and sign the honor pledge: *I pledge my honor that I have not violated the Honor Code during this examination.*

# Physical Constants and Conversion Factors

$$
\begin{aligned}
c &= 2.998 \times 10^{10}\ \mathrm{cm\,s^{-1}}\,, \\
\hbar &= 1.054 \times 10^{-27}\ \mathrm{erg\,s}\,, \\
k &= 1.380 \times 10^{-16}\ \mathrm{erg\,K^{-1}}\,, \\
e &= 4.803 \times 10^{-10}\ \mathrm{statcoulomb}\,, \\
N_0 &= 6.025 \times 10^{23}\ \mathrm{molecules\ \ mole^{-1}}\,, \\
m_{\mathrm{electron}} &= 9.108 \times 10^{-28}\ \mathrm{g}\,, \\
m_{\mathrm{proton}} &= 1.672 \times 10^{-24}\ \mathrm{g}\,, \\
m_{\mathrm{neutron}} &= 1.675 \times 10^{-24}\ \mathrm{g}\,, \\
m_{\mathrm{amu}} &= 1.660 \times 10^{-24}\ \mathrm{g}\,, \\
\mu_B &= 9.273 \times 10^{-21}\ \mathrm{erg\ \ Gauss^{-1}}\,, \\
G &= 6.673 \times 10^{-8}\ \mathrm{cm^3\,s^{-2}\ \ g^{-1}}\,.
\end{aligned}
$$

$$
\begin{aligned}
1\ \mathrm{atm} &= 1.013 \times 10^{6}\ \mathrm{dyne\,cm^{-2}}\,, \\
1\ \mathrm{eV} &= 1.602 \times 10^{-12}\ \mathrm{erg}\,, \\
1\ \mathrm{cal} &= 4.186 \times 10^{7}\ \mathrm{erg}\,.
\end{aligned}
$$

1. Two identical objects, A and B, are thermally and mechanically isolated from the rest of the world. Their initial temperatures are $\tau_A > \tau_B$. Each object has heat capacity $C$ (the same for both objects) which is independent of temperature.

(a) Suppose the objects are placed in thermal contact and allowed to come to thermal equilibrium. What is their final temperature? How much entropy is created in this process? How much work is done on the outside world in this process?

——— Solution ———

$\boxed{\text{No work}}$ is done—the objects are just in thermal contact. If energy $đQ$ is transfered from object A to B and the objects are allowed to come to equilibrium, the temperature change of object A is $d\tau_A = -đQ/C$ and the temperature change of object B is $d\tau_B = +đQ/C$. In other words, the temperatures of objects A and B change by equal and opposite amounts, so the final temperature is $\tau_f = \boxed{(\tau_A + \tau_B)/2}$. The change in entropy of the system is

$$\begin{aligned}
\Delta\,\sigma &= \int_{\tau_A}^{\tau_f} \frac{C\,d\tau}{\tau} + \int_{\tau_B}^{\tau_f} \frac{C\,d\tau}{\tau} \\
&= C \log\left(\frac{\tau_f^2}{\tau_A \tau_B}\right) \\
&= \boxed{C \log\left(\frac{(\tau_A + \tau_B)^2}{4\tau_A \tau_B}\right)}\ .
\end{aligned}$$

——— End Solution ———

(b) Instead, suppose objects A (temperature $\tau_A$) and B (temperature $\tau_B < \tau_A$) are used as the high and low temperature heat reservoirs of a heat engine. The engine extracts energy from object A (lowering its temperature), does work on the outside world, and dumps waste heat to object B (raising its temperature). When the temperatures of A and B are the same, the heat engine is in the same state as it started and the process is finished. Suppose this heat engine is the most efficient heat engine possible. In other words, it performs the maximum work possible. What is the final temperature of the objects? How much entropy is created in this process? How much work is done on the outside world in this process?

Solution

If we have the most efficient engine possible, it must be a reversible engine which means that the entropy created is $\boxed{\text{zero}}$. Any entropy extracted from object A must wind up in object B. This means

$$d\sigma = 0 = \frac{C\,d\tau_a}{\tau_a} + \frac{C\,d\tau_b}{\tau_b}\ ,$$

where $\tau_a$ and $\tau_b$ are the temperatures of object A and B. Integrating gives

$$\log\tau_a + \log\tau_b = \log(\tau_a\tau_b) = \text{Constant}\ ,$$

so $\tau_a\tau_b$ is constant in this process. Thus, $\tau_f = \boxed{\sqrt{\tau_A\tau_B}}$. When $\tau_a$ changes by $d\tau_a$, the work done on the outside world is

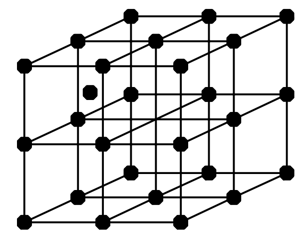
$$đW = -C\,d\tau_a - C\,d\tau_b = -C\left(1 - \frac{\tau_b}{\tau_a}\right)d\tau_a\ .$$

We could integrate this expression to find the total work done, but it's much easier to use the initial and final temperatures

$$\begin{aligned} W &= C\,(\tau_A - \tau_f) - C\,(\tau_f - \tau_B) \\ &= \boxed{C\,(\tau_A + \tau_B - 2\sqrt{\tau_A\tau_B})}\ . \end{aligned}$$

An interesting point to notice: if we plug $\tau_f = \sqrt{\tau_A\tau_B}$ into the expression for the entropy created that was derived in part (a), we find $d\sigma = 0$, as we should.

End Solution



2. A cubic crystal contains N atoms. The atoms can exist at lattice sites or an atom may find itself displaced from its normal site into the center of one of the 8 adjacent unit cells as suggested by the figure. When an atom is displaced, its energy is increased by $\epsilon > 0$ over its energy at the normal lattice site. Suppose the crystal is in equilibrium at temperature $\tau$. Assume $\tau \ll \epsilon$ so the chance that two atoms try to occupy the same cell is negligible. Calculate the partition function, the entropy, the energy (relative to the energy the crystal would have if all atoms were at their normal sites), and the number of atoms in displaced sites.

Solution

Note: $N$ is a large number. The 27 atom cube above is just to suggest how an atom is displaced into an interstitial space.

A single atom can exist in nine states: one with energy 0 at its lattice site, and 8 with energy $\epsilon$ in the adjacent cells. The partition function for one atom is

$$Z_1 = 1 + 8e^{-\epsilon/\tau} .$$

The partition function for the whole crystal is just the product of $N$ single atom functions (there is no $N!$ correction since the atoms are distinguishable—they leave behind an empty lattice site!),

$$Z = \left(1 + 8e^{-\epsilon/\tau}\right)^N .$$

The free energy is

$$F = -\tau \log Z = -N\tau \log\left(1 + 8e^{-\epsilon/\tau}\right) ,$$

$$\sigma = -\frac{\partial F}{\partial \tau} = N\left[\log\left(1 + 8e^{-\epsilon/\tau}\right) + \frac{8(\epsilon/\tau)e^{-\epsilon/\tau}}{1 + 8e^{-\epsilon/\tau}}\right]$$

$$U = F + \tau\sigma = \frac{8N\epsilon\, e^{-\epsilon/\tau}}{1 + 8e^{-\epsilon/\tau}} ,$$

Since each displaced atom has energy $\epsilon$, the number of such atoms is

$$n_{\text{displaced}} = U/\epsilon = \frac{8Ne^{-\epsilon/\tau}}{1 + 8e^{-\epsilon/\tau}} .$$

All the results above can be written for the case that $\tau \ll \epsilon$,

$$Z = \boxed{1 + 8Ne^{-\epsilon/\tau}} ,$$

$$F = -8N\tau e^{-\epsilon/\tau} ,$$

$$\sigma = \boxed{8N(\epsilon/\tau)e^{-\epsilon/\tau}} \ ,$$

$$U = \boxed{8N\epsilon\, e^{-\epsilon/\tau}} \ ,$$

$$n_{\text{displaced}} = \boxed{8Ne^{-\epsilon/\tau}} \ .$$

End Solution

3. Consider a spinless particle of mass $m$ bound to the origin by a quadratic potential $V = k(x^2 + y^2 + z^2)/2 - 3\hbar\omega/2$. In other words, a three dimensional harmonic oscillator. Its energies are $\epsilon(n_x, n_y, n_z) = (n_x + n_y + n_z)\hbar\omega$. (The zero point energy was removed in the expression for the potential!) The frequency $\omega^2 = k/m$ is the natural frequency of the oscillations and $n_x$, $n_y$, and $n_z$ are all integers greater than or equal to zero.

Some things to note: All energies are an integer times $\hbar\omega$ and the degeneracy increases as the energy increases. The energy of the ground state is 0 ($n_x = n_y = n_z = 0$) and the ground state is non-degenerate.

(a) Obtain an expression for $\mathcal{D}(\epsilon)$, the density of states, valid at energies much larger than $\hbar\omega$. (Recall that the number of states with energies less than $\epsilon$ is $\int_0^\epsilon \mathcal{D}(\epsilon')\,d\epsilon'$.) Hint: try a geometric approach and note that the energy is a linear function of the quantum numbers.

Solution

Consider the space of quantum numbers with axes $n_x$, $n_y$, and $n_z$. For energy $\epsilon$, all the states with energy less than or equal to $\epsilon$ are the lattice points in this space satisfying $n_x + n_y + n_z \le \epsilon/\hbar\omega = n$. All these lattice points are contained in the positive octant between the origin and the plane that intersects the coordinate axes at $n$. If one imagines the $n_x n_y$ plane as horizontal, then this contains the base of a triangular pyramid, and the altitude is along the $n_z$ axis. The volume of the pyramid is 1/3 the area of the base ($n^2/2$) times the altitude ($n$), so the number of states is $n^3/6 = (\epsilon/\hbar\omega)^3/6$. Differentiating to get the density of states:

$$\mathcal{D}(\epsilon) = \boxed{\frac{\epsilon^2}{2\hbar^3\omega^3}}\ .$$

End Solution

(b) Now suppose there are $N$ ($N$ is very large, the order of Avogadro's number, say) noninteracting identical particles in the harmonic oscillator potential just described. The particles are in equilibrium at temperature $\tau$ which is much less than $\hbar\omega$ so that almost all particles are in the ground state. (Remember, they're spinless, so they're bosons and there's no limit to how many particles may occupy a state!) What is the chemical potential in this low temperature limit?

Solution

The occupancy of the ground state is $1/(\exp(-\mu/\tau) - 1)$. Since the temperature is low enough that almost all $N$ particles are in the ground state, we must have $N = 1/(\exp(-\mu/\tau) - 1)$. Or, $\exp(-\mu/\tau) - 1 = 1/N$. Or $\boxed{\mu = -\tau/N}$.

End Solution

(c) This system has something like a Bose Einstein Condensation (BEC) such that the number in the ground state is large even for temperatures comparable to $\hbar\omega$. The number of particles in the ground state is

$$N_0 = N\left(1 - \left(\frac{\tau}{\tau_E}\right)^\alpha\right) ,$$

where $\tau_E$ is the "Einstein condensation temperature." Determine a numerical value for $\alpha$ and an expression for $\tau_E$. In deriving this expression, you may encounter an intractable integral. It is not necessary to evaluate this integral. Instead convert it to dimensionless form (that is, it does not contain $\hbar$, $\omega$, $\tau$, $\mu$ or $N$) and call its value $I$. Make suitable approximations!

Solution

The number of particles in excited states (to a good approximation) is

$$N_e = \int_0^\infty \frac{\mathcal{D}(\epsilon)\, d\epsilon}{e^{(\epsilon - \mu)/\tau} - 1}$$

We saw in part (b) that $\mu$ is small, so to a good approximation, we may ignore $\mu$ (except for the ground state which makes no contribution to the above integral!). Then the number in excited states is

$$\begin{aligned}
N_e &= \int_0^\infty \frac{\epsilon^2}{2\hbar^3\omega^3}\frac{1}{e^{\epsilon/\tau} - 1}\, d\epsilon \\
&\qquad \text{let } x = \epsilon/\tau \\
&= \frac{1}{2}\left(\frac{\tau}{\hbar\omega}\right)^3 \int_0^\infty \frac{x^2\, dx}{e^x - 1} \\
&\qquad \text{denote the integral by } I \\
&= \frac{I}{2}\left(\frac{\tau}{\hbar\omega}\right)^3 \\
&= N\left(\frac{\tau}{2^{1/3}N^{1/3}\hbar\omega/I^{1/3}}\right)^3 \\
&\qquad \text{let } \tau_E = 2^{1/3}N^{1/3}\hbar\omega/I^{1/3} \\
&= N\left(\frac{\tau}{\tau_E}\right)^3
\end{aligned}$$

Then the number in the ground state is

$$N_0 = N\left(1 - \left(\frac{\tau}{\tau_E}\right)^3\right)$$

and $\boxed{\alpha = 3}$ and $\boxed{\tau_E = 2^{1/3}N^{1/3}\hbar\omega/I^{1/3}}$ where $I$ is the integral defined above. As a point of interest, $I = 2.40$, so $\tau_E = 0.94N^{1/3}\hbar\omega$.

End Solution

4. A small spherical particle, radius $r = 1\mu = 10^{-4}\,\mathrm{cm}$ moves in room temperature ($T = 300\,\mathrm{K}$) water. The density of the particle is about the same as that of water, $\rho = 1\,\mathrm{g/cm^3}$, and the viscosity of water is about $\eta = 0.01\,\mathrm{poise} = 0.01\,\mathrm{g/s\,cm}$.

In all cases below, both an analytic expression and a numerical estimate are desired.

(a) Suppose the particle is moving with some initial velocity. Use dimensional analysis and what you know about viscosity to estimate the decay time for the particle's velocity.

Solution

The viscosity is the proportionality constant between the shear force per unit area and the velocity gradient. So the shear force is proportional to the area times the velocity gradient. The area must be roughly $\pi r^2$. The velocity gradient must be $v/r$. (That is, about one radius from the particle we expect that the fluid "doesn't know" there's a moving particle nearby.) The mass of the particle is $m = 4\pi r^3 \rho/3$. Using $F = ma$, we have

$$4\pi r^3 \rho \dot{v}/3 \approx -\pi r^2 \eta v/r$$

or

$$t \approx \left|\frac{v}{\dot{v}}\right| \approx \boxed{\frac{4r^2\rho}{3\eta}} .$$

Plugging in the numbers: $t \approx 1.33 \cdot (10^{-4})^2 \cdot 1/0.01\,\mathrm{s} = 1.3 \times 10^{-6}\mathrm{s} \approx \boxed{10^{-6}\,\mathrm{s}}$.

End Solution

(b) Assuming the particle is in equilibrium at temperature $T$, what is its typical velocity?

Solution

This is a case of equipartition. We expect that $mv^2/2 = 3kT/2$, so a typical velocity (the rms velocity) is $v \approx \boxed{\sqrt{3kT/m}}$.

Plugging in: $v = \sqrt{3 \cdot 1.38 \times 10^{-16} \cdot 300/(4\pi(10^{-4})^3 \cdot 1/3)}\,\mathrm{cm/s} = \boxed{0.17\,\mathrm{cm/s}}$.

End Solution

(c) About how far will the particle travel in one decay time?

Solution

$d \approx \boxed{vt} = 0.00000017\,\mathrm{cm} \approx \boxed{2 \times 10^{-7}\,\mathrm{cm}}$.

End Solution

(d) About how far will the particle travel in $M = 1$ million decay times? (Note: this question is asking for the typical distance between the particle's position at some time and its position a million decay times later.)

Solution

After one decay time, the velocity of the particle has been randomized. Therefore the particle does a random walk, with the step size from part (c) and a number of steps equal to $M$. The typical distance traveled is the distance per step times the square root of the number of steps, so $D = \boxed{d\sqrt{M}} = 0.0000002\sqrt{10^6}\,\text{cm} = \boxed{2 \times 10^{-4}\,\text{cm}}$

End Solution

5. During our studies of thermal physics, we've often considered an ideal gas. Sometimes we needed a model for departures from an ideal gas and we used a van der Waals gas. Of course, this isn't a perfect description of a real gas either, and lots of folks have invented modifications to the ideal gas equation of state that describe departures from ideal gas behavior. Here's Dieterici's equation of state for $N$ molecules of gas occupying a volume $V$ at pressure $p$ and temperature $\tau$,

$$p(V - Nb) = N\tau e^{-Na/\tau V} ,$$

where $a$ and $b$ are constants (not necessarily the same constants that occur in the van der Waals equation of state).

(a) Give brief statements of the physical significance of the constants $a$ and $b$.

Solution

The constant $b$ is similar to the constant $b$ that occurs in the van der Waals equation of state. It represents the volume occupied by a molecule. Molecules cannot be pushed so close that they would occupy a smaller volume, so $b$ represents a hard sphere repulsion at short distances.

When $b$ can be ignored ($V/N \gg b$), the constant $a$ reduces the pressure below what it would be for an ideal gas with the same number of molecules, temperature, and volume. Thus $a$ represents the effects of a long range attraction between molecules.

End Solution

(b) Construct a $p$-$V$ diagram and sketch some representative isotherms for this gas. Be sure to sketch the critical isotherm, at least one isotherm at a higher temperature than the critical isotherm and at least one isotherm at a lower temperature than the critical isotherm. Be sure to label the axes and the curves appropriately. Hints: you may want to use dimensionless variables—if so, be sure to specify the relation between your variables and $p$, $V$, and $\tau$. You may also want to consider part (c) while you do this part.

Solution

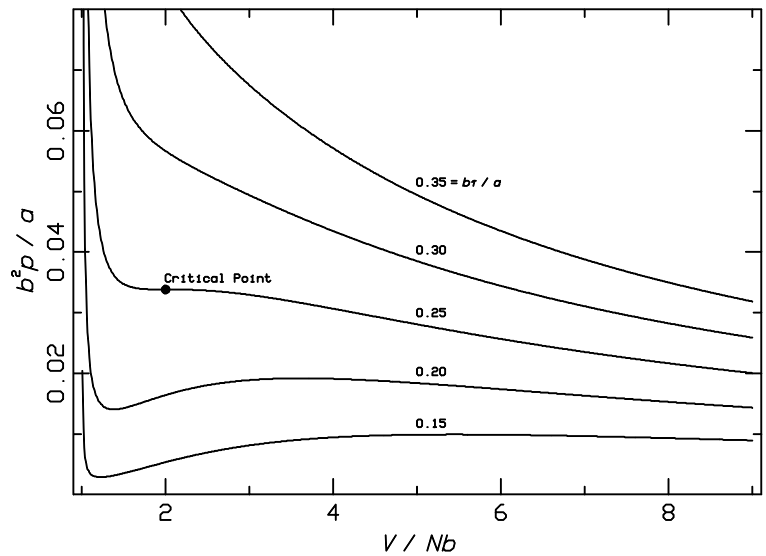
Dimensionless variables are

$$\hat{V} = \frac{V}{bN}\,, \quad \hat{\tau} = \frac{b\tau}{a}\,, \quad \hat{p} = \frac{b^2 p}{a}\,.$$

In these variables the equation of state can be written

$$\hat{p} = \frac{\hat{\tau}}{\hat{V} - 1}\, e^{-1/\hat{\tau}\hat{V}}\,.$$

The figure shows sketches of the isotherms. Note that the minimum $\hat{V}$ is 1 and even at low temperatures, the pressure takes off as $\hat{V} \to 1$.




End Solution

(c) Determine $p_c$, $V_c$, and $\tau_c$, that is, the pressure, volume, and temperature at the critical point.

Solution

We'll use the dimensionless variables defined in the solution for the previous part. The critical point is that point where $\partial\hat{p}/\partial\hat{V} = 0$ and $\partial^2\hat{p}/\partial\hat{V}^2 = 0$.

$$\frac{\partial \hat{p}}{\partial \hat{V}} = -\frac{\hat{\tau}}{(\hat{V}-1)^2} e^{-1/\hat{\tau}\hat{V}} + \frac{1}{\hat{V}^2(\hat{V}-1)} e^{-1/\hat{\tau}\hat{V}} = \frac{e^{-1/\hat{\tau}\hat{V}}}{\hat{V}-1}\left(-\frac{\hat{\tau}}{\hat{V}-1} + \frac{1}{\hat{V}^2}\right)\,.$$

$$\frac{\partial^2 \hat{p}}{\partial \hat{V}^2} = \left(-\frac{e^{-1/\hat{\tau}\hat{V}}}{(\hat{V}-1)^2} + \frac{e^{-1/\hat{\tau}\hat{V}}}{\hat{\tau}\hat{V}^2(\hat{V}-1)}\right)\left(-\frac{\hat{\tau}}{\hat{V}-1} + \frac{1}{\hat{V}^2}\right) + \frac{e^{-1/\hat{\tau}\hat{V}}}{\hat{V}-1}\left(\frac{\hat{\tau}}{(\hat{V}-1)^2} - \frac{2}{\hat{V}^3}\right)\,,$$

$$= \frac{e^{-1/\hat{\tau}\hat{V}}}{\hat{V}-1}\left(+\frac{2\hat{\tau}}{(\hat{V}-1)^2} - \frac{2}{\hat{V}^2(\hat{V}-1)} + \frac{1}{\hat{\tau}\hat{V}^4} - \frac{2}{\hat{V}^3}\right)\,.$$

Setting the first derivative to zero gives $\hat{\tau} = (\hat{V}-1)/\hat{V}^2$. Setting the second derivative to zero and substituting for $\hat{\tau}$ gives $\hat{V}_c = 2$. Then $\hat{\tau}_c = 1/4$ and $\hat{p}_c = e^{-2}/4$. This gives

$$V_c = \boxed{2bN}\,, \qquad \tau_c = \boxed{\frac{a}{4b}}\,, \qquad p_c = \boxed{\frac{a}{4b^2}e^{-2}}\,.$$

Even easier: set the first derivative to 0, solve for $\hat{V}$ and find $\hat{\tau}$ which gives a double root.

End Solution