Homework 6 Solutions

Problem 1: Kittel 7-2 (a) The method is the same as for the non-relativistic gas. A particle confined in a box of volume $L^3$ is described by a set of wavefunctions $\psi_n \sim \sin(nx\pi x/L) \sin(ny\pi y/L) \sin(nz\pi z/L)$, which corresponds to a state with wavenumber:

$$k_n = \frac{\pi}{L} \sqrt{n_x^2 + n_y^2 + n_z^2} = \frac{\pi}{L} n$$

The energy of this state comes from the dispersion relation:

$$\epsilon_n = p_n c = \frac{\hbar k_n c}{L} = \frac{\pi \hbar c}{L} n$$

Please note that $p = \hbar k$ is a definition which has nothing to do with “relativistic” vs “nonrelativistic”. The number of orbitals with energy less than some energy $\epsilon_n$ is given by:

$$N(<\epsilon_n) = 2 \cdot \frac{14\pi}{3} n^3 = \frac{\pi}{3} \frac{L^3}{\pi^3 \hbar^3 c^3} \epsilon_n^3$$

The factor 2 is due to the spin degeneracy. To find the Fermi energy, we set $N(<\epsilon_n)$ equal to the total number of fermions $N$:

$$N = \frac{\pi}{3} \frac{L^3}{\pi^3 \hbar^3 c^3} \epsilon_F^3$$

$$\Rightarrow \quad \epsilon_F = \left( \frac{3N\pi^3 \hbar^3 c^3}{L^3 \pi} \right)^{1/3} = \pi \hbar c \left( \frac{3N}{\pi L^3} \right)^{1/3}$$

(b) To obtain the density of states, we differentiate $N(<\epsilon)$:

$$n(\epsilon)d\epsilon = \frac{dN(<\epsilon)}{d\epsilon} = \frac{\pi}{3} \frac{L^3}{\pi^3 \hbar^3 c^3} \epsilon^2 d\epsilon$$

The ground state of the system is the zero temperature state. In this limit, the Fermi-Dirac distribution becomes:

$$f(\epsilon) = \begin{cases} 1 & \text{if } \epsilon < \epsilon_F \\ 0 & \text{if } \epsilon > \epsilon_F \end{cases}$$

Therefore, the ground state energy is given by:

$$U_0 = \int_0^{\epsilon_F} d\epsilon n(\epsilon) \epsilon = \frac{\pi}{3} \frac{L^3}{\pi^3 \hbar^3 c^3} \int_0^{\epsilon_F} d\epsilon \epsilon^3$$

$$= \frac{3N}{\epsilon_F^4}$$

$$= \frac{3}{4} N \epsilon_F$$
(c) In the neutron star calculation, discussed in pages 17-4 to 17-6 of the lecture notes, it was found that the Fermi energy for nucleons and electrons was approximately 86 MeV. The relativistic dispersion relation for a particle of rest mass $m$ is given by:

$$\epsilon = \sqrt{p^2c^2 + m^2c^4} - mc^2$$

This expression gives the energy of the particle beyond the energy required to create the particle. The energy required to create the particle, $mc^2$, is interpreted as a chemical potential, as in the lecture notes. The mass of a proton is about 938 MeV, much greater than the Fermi energy of protons, so the non-relativistic approximation ($\epsilon \approx \frac{p^2}{2m}$) is fine for the nucleons. However, the electron mass of 0.511 MeV is much less than the electron Fermi energy, suggesting the ultrarelativistic limit ($\epsilon \approx pc$) for the electrons.

As in the notes, we consider a neutron star as a Fermi gas of mainly neutrons, with a small concentration of electrons and protons. The concentrations are denoted by $n_e, n_p,$ and $n_n$. The argument is exactly the same as in the notes (refer to pages 17-5 to 17-6 for notation) except now the electron Fermi energy is given by:

$$\epsilon_{F,e} = \pi \hbar c \left( \frac{3n_e}{\pi} \right)^{1/3} = \pi \hbar c \left( \frac{3n}{\pi} \right)^{1/3}x^{1/3}$$

The energy balance equation becomes:

$$\epsilon_{F0} \left( \frac{n}{n_0} \right)^{2/3} (1 - x)^{2/3} = \epsilon_{F0} \left( \frac{n}{n_0} \right)^{2/3} \frac{m_n}{m_p} x^{2/3} + \pi \hbar c \left( \frac{3n}{\pi} \right)^{1/3} x^{1/3} - E$$

or

$$(1 - x)^{2/3} + \frac{E}{\epsilon_{F0}} \left( \frac{n}{n_0} \right)^{2/3} = \frac{m_n}{m_p} x^{2/3} + \pi \hbar c \left( \frac{3n}{\pi} \right)^{1/3} x^{1/3}$$

The assumption that $n_e, n_p << n_n$ implies that $n_0 \approx n$ and that $x << 1$. This approximation is consistent with setting $n = n_0$ in the above expression; ignoring $x$ relative to 1 on the left hand side; and ignoring the $x^{2/3}$ term relative to the $x^{1/3}$ term on the right hand side. Making the approximation:

$$1 + \frac{E}{\epsilon_{F0}} \approx \frac{\pi \hbar c}{\epsilon_{F0}} \left( \frac{3n_0}{\pi} \right)^{1/3} x^{1/3}$$

$$x^{1/3} = \frac{1 + \frac{E}{\epsilon_{F0}}}{\frac{\pi \hbar c}{\epsilon_{F0}} \left( \frac{3n_0}{\pi} \right)^{1/3}}$$

$$x = \left( \frac{1 + \frac{E}{\epsilon_{F0}}}{\frac{\pi \hbar c}{\epsilon_{F0}} \left( \frac{3n_0}{\pi} \right)^{1/3}} \right)^3$$

$$= \left( \frac{1 + \frac{0.783}{86}}{\frac{\pi (197 \times 10^{-13})}{86 \left( \frac{3(2.9 \times 10^{18})}{\pi} \right)^{1/3}}} \right)^3$$

$$\approx 8 \times 10^{-3}$$
So our small $x$ approximation is self-consistent, though $x$ is larger than in the non-relativistic case.

**Problem 2:** Kittel 7-5 (a) We are given that liquid $^3$He has density $\rho = 0.081$ g/cm$^3$ and the atoms obey Fermi statistics. We compute the concentration:

$$n = (0.081) \times \frac{6 \times 10^{23}}{3} \text{cm}^{-3} = 1.62 \times 10^{22} \text{cm}^{-3},$$

the Fermi energy:

$$\epsilon_F = \frac{\hbar^2}{2m} (3\pi^2 n)^{2/3} = \frac{(1.05 \times 10^{-34})^2}{(2)(3)(1.67 \times 10^{-27})} \left(3\pi^2 \cdot (1.62 \times 10^{28})\right)^{2/3} \text{J}$$

$$= 6.74 \times 10^{-23} \text{J}$$

$$= 4.2 \times 10^{-4} \text{eV},$$

the Fermi velocity:

$$v_F = \left(\frac{2\epsilon_F}{m}\right)^{1/2} = \left(\frac{2 \cdot (6.74 \times 10^{-23})}{3 \cdot (1.67 \times 10^{-27})}\right)^{1/2} \text{m/s} = 164 \text{m/s},$$

and the Fermi temperature:

$$T_F = \frac{\epsilon_F}{k_B} = \frac{6.74 \times 10^{-23}}{1.38 \times 10^{-23}} \text{K} = 4.9 \text{K}$$

(b) The heat capacity is given by:

$$C_V = \frac{\pi^2}{2T_F} (Nk_BT) \approx 1.006(Nk_BT)$$

where $T$ is the temperature in Kelvin. This value is roughly a factor of 3 lower than the experimental value, which indicates something else is going on.

**Problem 3:** Kittel 7-6 (a) The gravitational self-energy of the star should depend on $G$, $M$ and $R$. Dimensional analysis suggests the following estimate:

$$U \sim -\frac{GM^2}{R}$$

The negative sign is due to the attractive nature of the force. (b) The electrons in the star may be treated as a degenerate Fermi gas. The kinetic energy of the electrons may be estimated:

$$K \sim N\epsilon_F \sim N \frac{\hbar^2}{2m} (3\pi^2 n)^{2/3} \sim \frac{N\hbar^2}{m} \left(\frac{N}{R^3}\right)^{2/3} \sim \frac{\hbar^2}{mR^2} N^{5/3} \sim \frac{\hbar^2 M_H^{5/3}}{mR^2}$$

Here $N \sim (M/M_H)$ where $M$ is the mass of the star and $M_H$ is the mass of a proton; $m$ is the mass of an electron. We have assumed the star is neutral so
there are equal numbers of electrons as positive ions (which would have a mass on the order of a proton mass and account for the mass of the star).

(c) Kittel argues that the kinetic energy of the electrons (primarily due to electrons) and the gravitational energy of the ions should be of similar orders of magnitude. This implies:

$$\frac{GM^2}{R} \sim \frac{\hbar^2 M^{5/3}}{mR^2 M_H^{5/3}}$$

$$M^{1/3}R \sim \frac{\hbar^2}{mM_H^{5/3}G} = \frac{(1.05 \times 10^{34})^2}{(10^{-28})(1.67 \times 10^{-24})^{5/3}(6.6 \times 10^{-8})^{1/3}} \text{cm}$$

$$= 7 \times 10^{20} \text{g}^{1/3} \text{cm}$$

(d) If the mass of the white dwarf is that of the sun, then part (c) implies:

$$R \sim \frac{10^{20}}{(2 \times 10^{33})^{1/3}} \text{cm} \sim 8 \times 10^8 \text{cm}$$

$$\rho \sim \frac{M}{R^3} \sim \frac{2 \times 10^{33}}{(8 \times 10^8)^3} = 4 \times 10^6 \text{g/cm}^3$$

(e) If the Fermi gas is composed of neutrons instead of electrons, then the results are changed by the mass ratio:

$$(M^{1/3}R)_n \sim \left(\frac{m_n}{m_e}\right)(M^{1/3}R)_e \sim (10^{-3})(7 \times 10^{20})^{1/3} \text{cm} \sim 10^{18} \text{g}^{1/3} \text{cm}$$

If the neutron star had the mass of the sun, then:

$$R \sim \frac{10^{17}}{(2 \times 10^{33})^{1/3}} \times \left(\frac{10^{-3}}{10^2}\right) \text{km} \sim 8 \text{km}$$

**Problem 4: Kittel 7-8** In the regime $\tau < \tau_E$, the chemical potential is very small, $\mu \sim -\tau/N$. The approximation $\mu \approx 0$ becomes exact in the limit of large $N$. In this limit:

$$U = \int_0^\infty d\epsilon D(\epsilon)f(\epsilon, \tau)\epsilon$$

$$= \frac{V}{4\pi^2} \left(\frac{2M}{\hbar^2}\right)^{3/2} \int_0^\infty d\epsilon \frac{e^{3/2}}{e^{\epsilon/\tau} - 1}$$

$$= \frac{V}{4\pi^2} \left(\frac{2M}{\hbar^2}\right)^{3/2} \left(\int_0^\infty dx \frac{x^{3/2}}{e^x - 1}\right)^{5/2}$$

$$C_V = \left(\frac{\partial U}{\partial T}\right)_V = \frac{V}{4\pi^2} \left(\frac{2M}{\hbar^2}\right)^{3/2} \left(\int_0^\infty dx \frac{x^{3/2}}{e^x - 1}\right)^{5/2} \frac{5}{2} x^{3/2}$$
To get the entropy, we note that for constant volume, \( dU = \tau d\sigma \Rightarrow d\sigma = \frac{dU}{\tau} \). Therefore:

\[
\left( \frac{\partial \sigma}{\partial T} \right)_V = \frac{1}{\tau} \left( \frac{\partial U}{\partial T} \right)_V = \frac{V}{4\pi^2} \left( \frac{2M}{\hbar^2} \right)^{3/2} \left( \int_0^\infty dx \frac{x^{3/2}}{e^x - 1} \right)^{-\frac{5}{2}} \tau^{1/2}
\]

We integrate both sides of this equation with respect to temperature (keeping the volume constant) to get the entropy.

\[
\sigma(\tau) - \sigma(0) = \int_0^\tau \left( \frac{\partial \sigma}{\partial T} \right)_V d\tau = \frac{V}{4\pi^2} \left( \frac{2M}{\hbar^2} \right)^{3/2} \left( \int_0^\infty dx \frac{x^{3/2}}{e^x - 1} \right)^{-\frac{5}{2}} \tau^{1/2}
\]

In the case where the temperature \( \tau > \tau_E \), we can no longer take the chemical potential to be small. In this case, the energy is formally given by the expression:

\[
U = \int_0^\infty d\epsilon D(\epsilon) f(\epsilon, \tau, \mu) \epsilon \\
= \frac{V}{4\pi^2} \left( \frac{2M}{\hbar^2} \right)^{3/2} \left( \int_0^\infty dx \frac{x^{3/2}}{e^x - 1} \right)^{-\frac{3}{2}} \frac{\tau}{\tau^{1/2}}
\]

In order to obtain \( \mu \), we have to consider the particle number constraint:

\[
N = \int_0^\infty d\epsilon D(\epsilon) f(\epsilon, \tau, \mu) = \frac{V}{4\pi^2} \left( \frac{2M}{\hbar^2} \right)^{3/2} \left( \int_0^\infty dx \frac{e^{1/2}}{e^{(\epsilon-\mu)/\tau} - 1} \right)
\]

This gives an equation which, at least formally, may be inverted to get \( \mu \). Then, we may numerically integrate the previous expression to obtain the energy. Given the energy, we obtain the heat capacity and entropy as in the \( \tau < \tau_E \) case.

**Problem 5: Kittel 7-12** We showed in Homework 4, Problem 6 (Kittel 5-10), that if a system has average particle number \( < N > \), the mean squared fluctuation in the particle number is given by \( < (\Delta N)^2 > = \tau \frac{\partial < N >}{\partial \mu} \). In this problem, we are considering a single orbital so the average particle number is given by the Bose-Einstein distribution function:

\[
< N > = \frac{1}{e^{(\epsilon-\mu)/\tau} - 1}
\]

The rest is straightforward:

\[
< (\Delta N)^2 > = \tau \frac{\partial < N >}{\partial \mu} = \tau \left( \frac{e^{(\epsilon-\mu)/\tau}}{(e^{(\epsilon-\mu)/\tau} - 1)^2} \right) \frac{1}{\tau}
\]

\[
= \frac{1}{(e^{(\epsilon-\mu)/\tau} - 1)} + \frac{1}{(e^{(\epsilon-\mu)/\tau} - 1)^2}
\]

\[
= < N > + < N >^2 = < N > (1 + < N >)
\]
Problem 6 We are consider a gas of 2000 atoms of $^{87}$Rb. The observed Bose-Einstein transition occurs at the temperature $T_E = 170 \times 10^{-9}$K. In order to get the concentration, we use the formula for $T_E$ given on page 18-4 of the lecture notes:

$$
T_E = \frac{2\pi \hbar^2}{Mk_B} \left( \frac{N}{2.612V} \right)^{2/3}
$$

$$
\frac{N}{V} = 2.612 \left( \frac{M}{2\pi \hbar^2 k_B T_E} \right)^{3/2}
$$

$$
= 2.612 \left( \frac{87 \times 1.67 \times 10^{-27} \cdot 1.38 \times 10^{-23} \cdot 170 \times 10^{-9}}{2\pi(1.05 \times 10^{-34})^2} \right)^{3/2} \text{m}^{-3}
$$

$$
= 2.85 \times 10^{19} \text{m}^{-3}
$$

$$
= 2.85 \times 10^{13} \text{cm}^{-3}
$$

To obtain the density, we multiply this by the mass per atom:

$$
\rho = \frac{NM}{V} = (2.85 \times 10^{13})(87 \times 1.67 \times 10^{-24}) \frac{\text{g}}{\text{cm}^3} = 4.14 \times 10^{-9} \frac{\text{g}}{\text{cm}^3}
$$

Assuming a spherical sample, then number of particles is given by $N = (\frac{4}{3}\pi R^3)(N/V)$ so that:

$$
R = \left( \frac{3N}{4\pi(\frac{V}{N})} \right)^{1/3} = \left( \frac{3 \cdot 2000}{4\pi(2.85 \times 10^{13})} \right)^{1/3} \text{cm} = 2.56 \times 10^{-4} \text{cm}
$$