Homework 3 Solutions
Problem 1
(a) The technique is essentially that of Homework 2, problem 2. The situation is depicted in the figure:

Figure 1: The figure shows the system at time $t$. Photons with direction specified by the angles $(\theta, \phi)$ must be in the volume $dV$ in order to hit area $A$ by the time $t + dt$.

We will first determine the contribution to the pressure coming from photons of frequency $\omega$ and direction $(\theta, \phi)$. Here $\theta$ is the polar angle, i.e. the angle between the photon direction and the vector normal to the wall, and $\phi$ is the azimuthal angle. A single such photon hitting the wall will transfer the following amount of momentum to the wall:

$$dP = \frac{2\hbar\omega}{c} \cos \theta$$

The number of such photons that hit the wall in time $dt$ is the number of photons found within the volume $dV$. This number, times the momentum transferred by a single photon, gives the total momentum transfer from “($\omega, \theta, \phi$)-photons”:

$$P(\omega, \theta, \phi) d\omega d\Omega = \frac{1}{[\frac{\hbar\omega}{c}]^2} \left[ \frac{V\omega^2}{\pi^2c^3} d\omega \right] \left[ \frac{\sin \theta d\theta d\phi}{4\pi} \right] \left[ \frac{Ac(dt) \cos \theta}{V} \right] \left[ \frac{2\hbar\omega}{c} \cos \theta \right]$$

Here the first term is the number of photons in a particular mode of frequency $\omega$; the second factor is the number of modes in the volume $V$ with frequency between $\omega$ and $\omega + d\omega$; the third term is the fraction of photons with directions specified by the angles $(\theta, \phi)$; the fourth term is the probability that a random photon lies in volume $dV$; and the fifth term is the momentum transfer by a single photon. The pressure is the total momentum transferred (by photons of all directions and frequencies) per time per area. We get this by dividing the above expression by $Adt$ and integrating over frequency and angles. While the range of $\phi$ is from $[0, 2\pi]$, the range of $\theta$ is from $[0 : \pi/2]$ because $\theta$ in $[\pi/2 : \pi]$ refers to photons moving away from the area $A$. Therefore:

$$P = \int_0^\infty d\omega \int_0^{\pi/2} \sin \theta d\theta \int_0^{2\pi} d\phi P(\omega, \theta, \phi)$$
\[ \frac{\hbar}{2\pi^3c^3} \int_0^\infty d\omega \frac{\omega^3}{e^{\hbar\omega/\tau} - 1} \int_0^{\pi/2} d\theta \cos^2 \theta \sin \theta \int_0^{2\pi} d\phi = \frac{\hbar}{2\pi^3c^3} \int_0^\infty \frac{4\pi \tau x^3}{3e^x - 1} dx = \frac{1}{3} \frac{\pi^2 \tau^4}{15c^3\hbar^3} \]

where in the last line, we used the fact that \( \int_0^\infty \frac{dx}{x^3} = \frac{\pi^4}{6} \).

(b) We know that \( V \sim L^3 \) which immediately implies that \( \frac{dV}{V} = 3 \frac{dL}{L} \). We also know that for a given electromagnetic mode in a cavity (corresponding to a triplet of integers \( (n_x, n_y, n_z) \)), the frequency varies inversely with length scale: \( \omega \sim L^{-1} \). This is discussed on page 10-3 of the lecture notes. This implies \( \frac{d\omega}{\omega} = -\frac{dL}{L} \). We note that if we change the volume at constant entropy, then the number of photons in a given mode should be the same. Because this number is given by \( n_\omega = (e^{-\hbar\omega/\tau} - 1)^{-1} \), we have that \( \omega/\tau \) should be constant. In other words, \( \omega \sim \tau \) so \( \frac{d\omega}{\omega} = \frac{d\tau}{\tau} \). So we have shown that (where the last equality holds only for constant entropy):

\[ \frac{dV}{V} = 3 \frac{dL}{L} = -3 \frac{d\omega}{\omega} = -3 \frac{d\tau}{\tau} \]

We know that the energy of a gas of photons in a box of volume \( V \) and temperature \( \tau \) is:

\[ U = \frac{\pi^2}{15\hbar^3 c^3} V \tau^4 \]

We may then calculate the pressure at constant entropy using our derived differential relations:

\[ p = -\left( \frac{\partial U}{\partial V} \right)_\sigma = -\frac{\pi^2}{15c^3\hbar^3}(\tau^4 + 4V\tau^3 \left( \frac{\partial T}{\partial V} \right)_\sigma) \]

\[ = -\frac{\pi^2}{15c^3\hbar^3}(\tau^4 + 4V\tau^3 \left( \frac{-\tau}{3V} \right)) \]

\[ = \frac{\pi^2 \tau^4}{45c^3\hbar^3} \]

**Problem 2** The average energy of an oscillator of frequency \( \omega \) was worked out on page 9-1 of the lecture notes:

\[ \langle E \rangle = \frac{\hbar \omega}{e^{\hbar\omega/\tau} - 1} \]
The energy of a photon in the oscillator is $\hbar \omega$. So the number of photons in an oscillator of frequency $\omega$ is:

$$N_\omega = \frac{1}{e^{\hbar \omega / \tau} - 1}$$

Note that “oscillator” is a synonym for “mode” in this context. You should not think of the word “oscillator” as meaning a single particle vibrating (which is the PHY105 usage) but as a particular mode that can be excited some integral number of times. A “photon” is the name given to such an excitation when we are talking about cavity radiation, “phonon” is the name when we deal with crystal lattices, etc. To get the total number of photons, we sum over modes which amounts to integrating using the density of states (see pages 8-7 to 8-8 of the lecture notes).

$$N = \frac{V}{\pi^2 c^3} \int_0^\infty d\omega \frac{\omega^2}{e^{\hbar \omega / \tau} - 1} = \frac{V \tau^3}{\pi^2 c^3 h^3} \int_0^\infty dx \frac{x^2}{e^x - 1} \approx 2.404 \frac{V \tau^3}{\pi^2 c^3 h^3}$$

The entropy of the gas is given on page 9-5 of the lecture notes:

$$\sigma = \frac{4\pi^2}{45 h^3 c^3} V \tau^3$$

We see from this that:

$$\frac{\sigma}{N} \approx \frac{4\pi^4}{2.404 \times 45} \approx 3.6$$

**Problem 3** This problem is similar to Problem 1. Referring to the figure accompanying that problem, $A$ is the area of the hole. An escaping “$(\omega, \theta, \phi)$-photon” contributes an amount $\hbar \omega$ to the energy that flows out of $A$ in unit time. The energy flux density is then given by $\hbar \omega$ times the number of $(\omega, \theta, \phi)$-photons flowing out in time $dt$ divided by $A dt$:

$$j(\omega, \theta, \phi) d\omega d\Omega = \left[ \frac{1}{e^{\hbar \omega / \tau} - 1} \right] \left[ \frac{V \omega^2}{\pi^2 c^3} d\omega \right] \left[ \frac{\sin \theta d\theta d\phi}{4\pi} \right] \left[ \frac{A(dt)c \cos \theta}{V} \right] \left[ \frac{\hbar \omega}{A(dt)} \right]$$

$$= \frac{\hbar}{\pi^2 c^2} \left[ \frac{\omega^3}{e^{\hbar \omega / \tau} - 1} d\omega \right] \left[ \frac{\cos \theta \sin \theta d\theta d\phi}{4\pi} \right]$$

To find the total energy flux outwards, we must integrate over frequency and solid angle (where the $\theta$ integration is only from $[0 : \pi/2]$):

$$J = \int_0^\infty d\omega \int_0^{\pi/2} \sin \theta d\theta \int_0^{2\pi} d\phi j(\omega, \theta, \phi)$$
\[ \frac{\hbar}{4\pi^2c^2} \left[ \int_0^{\infty} \frac{\omega^3}{e^{\hbar\omega/\tau} - 1} d\omega \int_0^{\pi/2} \cos \theta \sin \theta d\theta \int_0^{2\pi} d\phi \right] = \frac{\hbar}{4\pi^2c^2} \left[ \frac{\pi^4r^4}{15\hbar^4} \right] = \frac{\pi^2r^4}{60\hbar^3c^3} = \frac{1}{4Vc} \]

**Problem 4**

(a) An order of magnitude estimate is given by

\[ V \sim -\frac{GM^2}{R} = -\frac{6.6 \times 10^{-8} \times (2 \times 10^{33})^2}{7 \times 10^{10}} \text{ ergs} \sim -3.78 \times 10^{48} \text{ ergs.} \]

(b) The virial theorem says \( K = -\frac{1}{2}V \). A thermodynamic estimate for the kinetic energy is \( K \sim N k_b T \) where \( N \) is the number of particles. So an estimate of the temperature of the sun is:

\[ T \sim \frac{K}{Nk_b} \sim \frac{0.5 \times 3.78 \times 10^{48}}{10^{57} \times 1.38 \times 10^{-16}} \sim 1.4 \times 10^7 \text{ degrees K} \]

**Problem 5**

(a) A given oscillator of frequency \( \omega_n \) has energy levels given by \( E_s = \hbar \omega_n \).

The partition function for the oscillator is easily found:

\[ Z_n = \sum_{s=0}^{\infty} e^{-\hbar \omega_n / \tau} = \frac{1}{1 - e^{-\hbar \omega_n / \tau}} \]

Each oscillator is independent and distinguishable so that the total partition function is merely the product of the individual partition functions:

\[ Z = \prod_n Z_n = \prod_n \frac{1}{1 - e^{-\hbar \omega_n / \tau}} \]

(b)

\[ F = -\tau \log Z = \tau \sum_n \log [1 - e^{-\hbar \omega_n / \tau}] \]

\[ = \frac{V}{\pi^2c^3} \int_0^\infty \omega^2 \log [1 - e^{-\hbar \omega / \tau}] d\omega \]

\[ = \frac{V\tau^4}{\pi^2c^3\hbar^4} \int_0^\infty x^2 \log [1 - e^{-x}] dx \]

\[ = \frac{V\tau^4}{\pi^2c^3\hbar^4} \left( \frac{x^3}{3} \log (1 - e^{-x}) \right)_{0}^{\infty} - \frac{1}{3} \int_0^\infty \frac{x^3}{e^x - 1} dx \]

\[ = \frac{\pi^2V\tau^4}{45\hbar^3c^3} \]

where in the last step, we used \( \int_0^\infty \frac{x^3}{e^x - 1} dx = \frac{\pi^4}{15} \).

**Problem 6**

(a) Kittel problem 4.8 The figure shows three black planes with the associated heat fluxes. In region 1, the net heat flux (positive direction being rightward) is \( J_u - J_m \) and in region 2, the net flux is \( J_m - J_l \). The equilibrium
Figure 2: Three black planes are at equilibrium when the temperatures are $T_u, T_m,$ and $T_l$ respectively. The corresponding fluxes are given by the Stefan-Boltzmann blackbody radiation law, i.e. $|J_u| = \sigma B T_u^4$, etc.

condition is that the heat flux flowing into the middle plate is equal to the heat flux leaving the plate. Therefore,

$$J_u - J_m = J_m - J_l$$

$$\Rightarrow T_m^4 = \frac{1}{2}(T_u^4 + T_l^4)$$

$$\Rightarrow J_u - J_m = J_m - J_l = \frac{1}{2} \sigma B (T_u^4 - T_l^4)$$

So the presence of the middle plate ("heat shield") cuts the net energy flux in the vacuum space by half.

(b) Kittel problem 4.19 The figure shows two black sheets with a gray sheet in between. The gray sheet has absorptivity $a$, emissivity $e$, and reflectivity $r = 1 - a$. The equilibrium condition is that $a = e$ and that the net flux into

Figure 3: Three sheets are at equilibrium when the temperatures are $T_u, T_m,$ and $T_l$ respectively. The corresponding fluxes are given by the Stefan-Boltzmann blackbody radiation law, i.e. $|J_u| = e \sigma B T_u^4$, where $e = 1$ for the black sheets. Since the middle sheet is gray, it reflects some portion of the heat flux from the outer plates.

the middle sheet is zero:

$$(1 - r)J_u - J_m = J_m - (1 - r)J_l$$
\[ J_m = (1 - r)(J_u + J_l) \]
\[ (1 - r)J_u - J_m = (1 - r)\frac{1}{2}(J_u - J_l) \]

which gives the desired result that the net energy flux density is \((1 - r)\) times that where the middle sheet were black.

**Problem 7** The energy of a crystal is given on page 10-5 of the lecture notes:

\[ U = \frac{3V}{2\pi^2 h^3 v^3} r^4 \int_0^{\theta/T} \frac{x^3 dx}{e^x - 1} \]

In the limit \(T \gg \theta\), the upper limit of integration becomes much less than unity. In this case, the exponential is well approximated by the first few terms of its Taylor series throughout the range of integration:

\[
\begin{align*}
U & \approx \frac{3V}{2\pi^2 h^3 v^3} r^4 \int_0^{\theta/T} x^3 dx \\
& \approx \frac{3V}{2\pi^2 h^3 v^3} r^4 \int_0^{\theta/T} \frac{x^2 dx}{1 + \frac{1}{2} x + \frac{1}{6} x^2 + \cdots} \\
& \approx \frac{3V}{2\pi^2 h^3 v^3} r^4 \int_0^{\theta/T} x^2 \left(1 + \left(-\frac{1}{2} x - \frac{1}{6} x^2 + \cdots\right) + \left(-\frac{1}{2} x - \frac{1}{6} x^2 + \cdots\right)^2 + \left(-\frac{1}{2} x - \frac{1}{6} x^2 + \cdots\right)^3 + \cdots\right) dx \\
& \approx \frac{3V}{2\pi^2 h^3 v^3} r^4 \int_0^{\theta/T} x^2 \left(1 - \frac{1}{2} x - \frac{1}{6} x^2 + \frac{1}{4} x^2 + \cdots\right) dx \\
& = \frac{3V}{2\pi^2 h^3 v^3} r^4 \int_0^{\theta/T} x^2 \left(1 - \frac{1}{12} x^2 + \cdots\right) dx \\
& = \frac{3V}{2\pi^2 h^3 v^3} r^4 T^4 \left(\frac{1}{3} \frac{\theta}{T}\right)^3 - \frac{1}{8} \frac{\theta}{T} \left(\frac{\theta}{T}^3 + \frac{1}{60} \frac{\theta}{T}\right) + \cdots \\
& = U_1 + U_2 + U_3 + \cdots
\end{align*}
\]

The leading term is:

\[
U_1 = \frac{3V k_b^4}{2\pi^2 h^3 v^3} T^4 \frac{1}{3} \left(\frac{\theta}{T}\right)^3 \\
= \frac{3V k_b^4}{2\pi^2 h^3 v^3} T \frac{1}{3} \left(\frac{h^3 v^3}{k_b^3}\right) \frac{6\pi^2 N}{V} \\
= 3N k_b T
\]

where we have used \(\theta = \frac{h v}{k_b} \left(\frac{6\pi^2 N}{V}\right)^{1/3}\). The second term \(U_2\) is temperature independent so does not contribute to the specific heat. The third term is:

\[
U_3 = \frac{3V k_b^4}{2\pi^2 h^3 v^3} T^4 \frac{1}{60} \left(\frac{\theta}{T}\right)^5 \\
= \frac{3V k_b^4}{2\pi^2 h^3 v^3} \frac{h^3 v^3}{k_b^3} \frac{6\pi^2 N}{V} \frac{1}{60} \left(\frac{\theta}{T}\right)^5 \\
= 3N k_b \frac{\theta^2}{20} \left(\frac{\theta}{T}\right)
\]
So the specific heat, to leading order in temperature, is:

\[ C_v = 3Nk_b(1 - \frac{1}{20}(\frac{\theta}{T})^2) \]

When \( \theta = T \), this implies a \( C_v/n \) (per mol) of 23.70\( \frac{1}{\text{mol K}} \) which is within 0.2% of the actual value tabulated in Table 4.2 of Kittel.

**Problem 8**

(a) The only sound waves in liquid \(^4\)He are longitudinal, i.e. there is only 1 polarization instead of 3. The difference this makes in the Debye temperature may be seen by following the derivation in the lecture notes pages 10-2 to 10-4. The final result:

\[
\theta = \left( \frac{18\pi^2 N}{V} \right)^{1/3} \frac{\hbar v}{k_b} = (18\pi^2 0.145 \times 6.02 \times 10^{23} \times \frac{1}{4})^{1/3} \frac{1.05 \times 10^{-27} \times 2.383 \times 10^4}{1.38 \times 10^{-16}} K \approx 28.5 K
\]

(b) \[
\frac{C_v}{M} = \frac{12\pi^4 N_A k_b}{5m_{He} \theta^3} T^3 = 0.0209 \times T^3
\]

which seems fairly close to the experimental value.