PHYSICS DEPARTMENT, PRINCETON UNIVERSITY

PHYSICS 301 MIDTERM EXAMINATION

October 22, 2003, 10:00–10:50 am, Jadwin A06

SOLUTIONS

This exam contains two problems. Work both problems. The problems count equally although one might be harder than the other. Do all the work you want graded in the separate exam books.

Write legibly. If I can’t read it, it doesn’t count!

Put your name on all exam books that you hand in. (Only one should be necessary!!!) On the first exam book, rewrite and sign the honor pledge: *I pledge my honor that I have not violated the Honor Code during this examination.*
1. Consider a system of $N$ spin 1 particles with magnetic moments. These spins are in a magnetic field so that each spin has magnetic energy $-E$ when its magnetic moment is aligned with the field, energy $+E$ when anti-aligned and energy 0 when neither anti-aligned nor aligned (perpendicular). $E > 0$. There are weak interactions which keep the spins in equilibrium at temperature $\tau$. Otherwise we can ignore interactions of the spins with each other and anything else but the magnetic field. (For example, take the particles to be at fixed positions so any thermal effects due to the center of mass motions of the particles can be ignored, etc.)

(a) Without elaborate calculation, determine the energy, $U$, and the entropy $\sigma$ of this system at very low temperatures ($\tau \to 0$). Also, without elaborate calculation, determine the energy and entropy at very high temperatures ($\tau \to \infty$).

\begin{solution}
At low temperatures, thermal effects are unimportant, so all the magnets are aligned, the energy is $U(\tau = 0) = -NE$ and there is but a single state, so $\sigma(\tau = 0) = 0$. At very high temperatures, the spins must have completely random alignments, with one third aligned, one third anti-aligned and one third perpendicular. Thus $U(\tau = \infty) = 0$. Since there are three states for each spin and each state is equally probable, the total number of states is $3^N$ and $\sigma(\tau = \infty) = N \log 3$.
\end{solution}

(b) What is the partition function of this system at temperature $\tau$? Hint: there’s an easy way and a hard way to do this.

\begin{solution}
Consider a single spin. Its partition function is the sum of the Boltzmann factors for each state
\[ Z_1 = e^{+E/\tau} + 1 + e^{-E/\tau} = 1 + 2 \cosh(\frac{E}{\tau}) . \]
Since we have $N$ non-interacting identical systems, the partition function of the system is the product of the individual partition functions. There is no $N!$ overcounting correction (the spins are at fixed locations). So
\[ Z = (1 + 2 \cosh(\frac{E}{\tau}))^N . \]
\end{solution}

(c) Find the free energy, entropy and energy of this system at temperature $\tau$.

\begin{solution}
The free energy is $F = -\tau \log Z = -N \tau \log(1 + 2 \cosh(\frac{E}{\tau}))$. The entropy is
\[ \sigma = -\left( \frac{\partial F}{\partial \tau} \right)_{V,N} = N \log(1 + 2 \cosh(\frac{E}{\tau})) - N \frac{E}{\tau} \frac{2 \sinh(\frac{E}{\tau})}{1 + 2 \cosh(\frac{E}{\tau})} . \]
\end{solution}
and the energy is
\[ U = F + \tau \sigma = -NE \frac{2\sinh(E/\tau)}{1 + 2\cosh(E/\tau)}. \]

Note that the expressions for the energy and entropy give the same values in the limits \( \tau \to 0 \) and \( \tau \to \infty \) as in part (a). The limits for entropy when \( \tau \to 0 \) are a bit tricky to calculate!

End Solution

Consider a membrane (a two dimensional surface like a drum head). We are interested in the low temperature thermal energy content and heat capacity of this membrane due to vibrational modes in which the displacement of the surface is perpendicular to the surface (also like a drum!). For convenience, imagine that the membrane is a square of side \( L \) with area \( A = L \times L \) and it is fixed at the edges. The speed of waves on the membrane is \( v \) which we take to be independent of frequency.

(a) For a standing wave mode of frequency \( \omega \), what is the average energy when the mode is in equilibrium at temperature \( \tau \). Hint: each standing wave mode is a harmonic oscillator with discrete energy levels. We can call the quanta of these vibrations “drumons” (bad joke)!

Solution

You should probably remember the expression for the average energy of a harmonic oscillator. If you don’t, it’s straightforward to work out. The partition function is
\[ Z = \sum_{n=0}^{\infty} e^{-n\hbar\omega/\tau} = \frac{1}{1 - e^{-\hbar\omega/\tau}}. \]

Then the average energy is
\[ \langle E \rangle = \tau^2 \frac{\partial}{\partial\tau} \log Z = \frac{\hbar\omega}{e^{\hbar\omega/\tau} - 1}. \]

End Solution

(b) How many modes \( n(\omega) \) are there between frequencies \( \omega \) and \( \omega + d\omega \)?

Solution

Since the membrane is fixed at the edge, the amplitude must be zero there, and the components of the wavenumber must be such that an integral number of half wavelengths fit in the length \( L \). A typical standing wave looks like
\[ z(x, y, t) = z_0 \cos \omega t \sin \frac{\pi n_x x}{L} \sin \frac{\pi n_y y}{L}. \]
Then

\[ \omega = \nu k = v \sqrt{k_x^2 + k_y^2} = v \sqrt{\left(\frac{\pi n_x}{L}\right)^2 + \left(\frac{\pi n_y}{L}\right)^2}, \]

or

\[ \left(\frac{L\omega}{\pi v}\right)^2 = n_x^2 + n_y^2. \]

The number of modes with frequencies less than \( \omega \) is the area of a quarter circle with radius \( L\omega/\pi v \) (there's a mode at each lattice point in the \( n_x, n_y \) plane). Also note that there is only one polarization for each mode.

\[ N(<\omega) = \int_0^\omega n(\omega') d\omega' = \frac{\pi}{4} \left(\frac{L\omega}{\pi v}\right)^2. \]

To get \( n(\omega) \) we differentiate the above expression with respect to \( \omega \).

\[ n(\omega) = \frac{d}{d\omega} \frac{\pi}{4} \left(\frac{L\omega}{\pi v}\right)^2 = \frac{A\omega}{2\pi v^2}. \]

\[ \text{End Solution} \]

(c) What are the low temperature thermal energy content and heat capacity of the membrane? You may come up with an integral that's non-trivial. If so, put it in dimensionless form and call its value \( I \).

\[ \text{Solution} \]

We know the average energy per mode and the number of modes, so we just need to add them all up. Since we are at low temperatures, we will run out of excited modes before we run out of modes and we can take the upper limit of integration below to be \( \infty \):

\[ U = \int_0^\infty \langle E \rangle n(\omega) d\omega, \]

\[ = \int_0^\infty \frac{\hbar \omega}{e^{\hbar \omega/\tau} - 1} \frac{A\omega}{2\pi v^2} d\omega \]

change variables to \( x = \hbar \omega/\tau \)

\[ = \frac{A\tau^3}{2\pi^2 v^2} \int_0^\infty \frac{x^2 dx}{e^x - 1}, \]

represent the integral by \( I \)

\[ = \frac{IA\tau^3}{2\pi^2 v^2}. \]

Copyright © 2003, Princeton University Physics Department, Edward J. Groth
The low temperature heat capacity (at constant area) is then

\[ C_A = \left( \frac{\partial U}{\partial \tau} \right)_A = \frac{3IA\tau^2}{2\pi\hbar^2v^2}. \]

As a point of interest, \( I = 2.4041 \ldots \)