This exam contains five problems. Work any three of the five problems. All problems count equally although some are harder than others. Do all the work you want graded in the separate exam books. Indicate clearly which three problems you have worked and want graded. I will only grade three problems. If you hand in more than three problems without indicating which three are to be graded, I will grade the first three, only!

Write legibly. If I can’t read it, it doesn’t count!

Put your name on all exam books that you hand in. (Only one should be necessary!!!) On the first exam book, rewrite and sign the honor pledge: I pledge my honor that I have not violated the Honor Code during this examination.
Physical Constants and Conversion Factors

\[ c = 2.998 \times 10^{10} \text{ cm s}^{-1}, \]
\[ h = 1.054 \times 10^{-27} \text{ erg s}, \]
\[ k = 1.380 \times 10^{-16} \text{ erg K}^{-1}, \]
\[ e = 4.803 \times 10^{-10} \text{ statcoulomb}, \]
\[ N_0 = 6.025 \times 10^{23} \text{ molecules mole}^{-1}, \]
\[ m_{\text{electron}} = 9.108 \times 10^{-28} \text{ g}, \]
\[ m_{\text{proton}} = 1.672 \times 10^{-24} \text{ g}, \]
\[ m_{\text{neutron}} = 1.675 \times 10^{-24} \text{ g}, \]
\[ m_{\text{amu}} = 1.660 \times 10^{-24} \text{ g}, \]
\[ \mu_B = 9.273 \times 10^{-21} \text{ erg Gauss}^{-1}, \]
\[ G = 6.673 \times 10^{-8} \text{ cm}^3 \text{ s}^{-2} \text{ g}^{-1}. \]

\[ 1 \text{ atm} = 1.013 \times 10^6 \text{ dyne cm}^{-2}, \]
\[ 1 \text{ eV} = 1.602 \times 10^{-12} \text{ erg}, \]
\[ 1 \text{ cal} = 4.186 \times 10^7 \text{ erg}. \]
1. Two identical objects, A and B, are thermally and mechanically isolated from the rest of the world. Their initial temperatures are $\tau_A > \tau_B$. Each object has heat capacity $C$ (the same for both objects) which is independent of temperature.

(a) Suppose the objects are placed in thermal contact and allowed to come to thermal equilibrium. What is their final temperature? How much entropy is created in this process? How much work is done on the outside world in this process?

(b) Instead, suppose objects A (temperature $\tau_A$) and B (temperature $\tau_B < \tau_A$) are used as the high and low temperature heat reservoirs of a heat engine. The engine extracts energy from object A (lowering its temperature), does work on the outside world, and dumps waste heat to object B (raising its temperature). When the temperatures of A and B are the same, the heat engine is in the same state as it started and the process is finished. Suppose this heat engine is the most efficient heat engine possible. In other words, it performs the maximum work possible. What is the final temperature of the objects? How much entropy is created in this process? How much work is done on the outside world in this process?
2. Consider waves on a liquid surface where the restoring force is produced by surface tension. Assume there is a single polarization and the dispersion relation is

\[ \omega^2 = \frac{\gamma}{\rho} k^3, \]

where \( \gamma \) is the surface tension of the liquid, \( \rho \) is its density, \( \omega \) is the frequency of the waves and \( k \) is the wavenumber of the waves. Our goal is to find the contribution of these waves to the low temperature heat capacity of the liquid.

(a) If the surface is in equilibrium at temperature \( \tau \), what is the average energy of a wave with frequency \( \omega \)? (Ignore the \( \hbar \omega/2 \) zero point energy.)

(b) Suppose the surface is a square of side \( L \) and area \( A = L^2 \). How many modes are there with wavenumbers between \( k \) and \( k + dk \)?

(c) At low temperatures what are the energy and heat capacity of these surface waves? You will come up with an integral that can’t be done easily. Explain why it’s OK to set the upper limit to \( \infty \). Having done this, convert the integral to dimensionless form and call its value \( I \).
3. The normal boiling point (that is, at $p = 1$ atm) of mercury is $T = 630$ K. At this pressure and temperature, mercury vapor may be treated as a monatomic ideal gas. The latent heat of vaporization is $L = 5.93 \times 10^{11}$ erg mole$^{-1}$. The atomic weight of mercury is 200.6 amu.

(a) Suppose that the latent heat of vaporization is constant, that mercury vapor may be treated as an ideal gas over the range of interest, and that the specific volume of liquid mercury is negligible compared with the specific volume of mercury vapor. Estimate the vapor pressure of mercury at $T = 300$ K (roughly room temperature).

(b) What is the entropy of the liquid at 630 K? Hint: You may need to know the entropy of the vapor. In case you don’t remember the expression, you might want to work it out. Remember that the number of states in $d^3r$ and $d^3p$ is $dn = d^3r \, d^3p/(2\pi\hbar)^3$. You should also remember that for large $N$, $\log N! \rightarrow N \log N - N$. 
4. Consider a line of $N+1$ atoms. Each atom interacts only with its nearest neighbors, so there are $N$ interactions. $N$ is very large—we can always neglect 1 when compared to $N$. Also, $\log N! \approx N \log N - N$. There are two kinds of atoms on the line, type $A$ and type $B$. The interaction energy for an $AA$ pair is $-\epsilon$; for a $BB$ pair, $-\epsilon$; and for an $AB$ pair, $-\epsilon/2$, where $\epsilon > 0$. In other words, when the two atoms in a bond are the same, they are twice as strongly bound as when they are different. The whole line of atoms is in equilibrium at temperature $\tau$. Exchanging energy with the heat bath means that atoms must change positions. For example, if there is a configuration, $ABAB$, then the three bonds in this configuration have a total energy $-3\epsilon/2$. If the inner two atoms swap positions to give $AABB$, the three bonds have a total energy of $-5\epsilon/2$ and energy $\epsilon$ was given to the heat bath.

Let $x$ denote the fraction of $B$ atoms, so if there are $N_A$ and $N_B$ atoms of types $A$ and $B$, respectively, then $N_A = (1 - x)N$ and $N_B = xN$, and $0 \leq x \leq 1$.

(a) Suppose the temperature is high enough that the atoms of each type are randomly and independently distributed along the line (in other words, it is a homogeneous mixture). Obtain expressions for the energy, the entropy, and the Helmholtz free energy as a function of $\epsilon$, $\tau$, $N$ and $x$. Be sure to use the fact that $N$ is large to simplify your answers.

(b) Suppose the temperature is zero! What is the configuration of atoms in this case? What is the ground state energy? What is the entropy? What is the free energy? Once again, use the fact that $N$ is very large to simplify your answer.

(c) In general, the system may be a homogeneous mixture like that in part (a) or it may be more like the configuration in part (b). Suppose $x = 1/2$ and determine the transition temperature which separates the (b) like state from the (a) like state. Hint: consider, for the homogeneous mixture case, the plot of free energy versus $x$ at a given temperature.
5. Recall that a definition for the entropy is

$$\sigma = -\sum_i p_i \log p_i,$$

where $p_i$ is the probability that the system is in state $i$ and the sum is over all states which satisfy whatever constraints are placed on the system. Note that this definition handles zero probability states and also produces an additive entropy when two non-interacting systems are combined. In all cases, suppose the volume of the system is fixed. The number of particles in each state is $N_i$ and the energy of each state is $E_i$.

(a) Suppose the number of particles in the system is fixed and the energy is fixed. What are the probabilities that maximize the entropy? Of course, the sum will be include only those states which have the correct number of particles and energy. (Or, equivalently, the probability for a state with the wrong number of particles or energy or both is 0.) In this and subsequent parts, you may have to introduce some auxiliary “constants.” Be sure to identify or give a physical interpretation for each such constant. Hint: be sure to use the fact that the system is in some state: $\sum_i p_i = 1$.

(b) Now suppose the number of particles is fixed, but the system is in equilibrium with a heat bath such that its average energy is $E$. What are the probabilities that maximize the entropy? As before, be sure to identify or give a physical interpretation for any constants you introduce.

(c) Now suppose the system is in equilibrium with a heat bath and a “particle bath” (a reservoir with which it can exchange particles) such that the average energy is $E$ and the average number of particles in the system is $N$. Now what are the probabilities that maximize the entropy? As before, be sure to identify or give a physical interpretation for any constants you introduce.