Physics 301 Problem Set 9 Solutions

Problem 1. At zero temperature, there is no entropy contribution and the free energy is equal to the energy \( f(x) = u(x) \). The fact that there is a finite residual solubility of \(^3\text{He}\) in \(^4\text{He}\) means then that the curve \( u(x) \) has a local minimum at \( x \approx 0.94 \), and it curves slightly upwards after that. On the left, the minimum is at \( x = 0 \). Between the two points, it has a local maximum as usual. When the \( \text{He} \) mixture was discussed, we ignored the interaction energies completely - this has to be taken into account when the fraction of \(^3\text{He}\) is small.

Problem 2. For either of the phases, the non-mixing parts of the free energy is a linear function \( f_0(x) = f_0(0) + xf'_0(0) \) for small \( x \). The mixing entropy (per molecule) is \( \sigma_m = -(1-x)\log(1-x) + x\log x \). If we ignore the mixing energy (or consider it to be part of the linear approximation, then the full free energy is \( f(x) = f_0(0) + xf'_0(0) + \tau[(1-x)\log(1-x) + x\log x] \). The minimum of this \( \partial f/\partial x = 0 \) is at \( x = \exp(-f'_0/\tau) \). The fact that the liquid mixture is in equilibrium with the solid mixture means that \( x \) is between the minima of the solid and liquid free energy curves. The ratio of the concentrations in the solid and liquid phases is then \( k = x_s/x_l = \exp(-(f'_0_s-f'_0_l)/\tau) \). For the values given, \( f'_0_s-f'_0_l = 1eV \), and \( \tau = 1000k_B = (1000/300 \times 40) \approx 1/12eV \) which gives \( k = e^{-12} \approx 6 \times 10^{-6} \).

Problem 3. We have a finite amount of gold on a (infinite) crystal of silicon. At a temperature above the melting point of the eutectic, the composition will be therefore a lot of pure silicon and a little bit of homogeneous liquid with a certain ratio of elements which we can read off from the graph as the point of intersection of the horizontal line at the given temperature and the right ’wing’ of the curve. (All the gold will have melted into this liquid).

Now we assume that the diffusion only occurs in the vertical direction. The height of the liquid layer is then given (in Angstroms) by \( 1000n_{Au} = xn_{liq} \) where \( x \) is the ratio of constituents and \( n \) is the number density (the equation equates the number of gold atoms before and after the diffusion). \( n_{liq} \), the density of the homogeneous liquid is given by \( 1/n_{liq} = x/n_{Au} + (1-x)/n_{Si} \). The height of the liquid layer is then \( h = 1000n_{Au}/xn_{liq} = 1000(1 + (1-x)n_{Au}/xn_{Si}) \). We are given the mass densities \( \rho_{Au} = 19.3g/cc \) and \( \rho_{Si} = 2.33g/cc \). The number densities are then \( n_{Au} = 5.91 \times 10^{22}/cc \) and \( n_{Si} = 4.99 \times 10^{22}/cc. \)
At 400°C, the ratio $x = 0.31$ which gives $h \approx 3640\,\text{Å}$, and at 800°C, the ratio $x = 0.43$ which gives $h \approx 2570\,\text{Å}$.

**Problem 4.** The liquefaction coefficient is $\lambda = (H_{\text{out}} - H_{\text{in}})/(H_{\text{out}} - H_{\text{liq}})$. The denominator is the sum of the latent heat and the difference in enthalpies of the gas in the out and in stages. In the ideal gas approximation, this is $H_{\text{out}} - H_{\text{in}} \approx \Delta H + (5/2)N(\tau_{\text{in}} - \tau_{\text{liq}})$. The numerator for a van der Waals gas is $H_{\text{out}} - H_{\text{in}} = (5/2)N(\tau_{\text{out}} - \tau_{\text{in}}) + (N^2/V_{\text{out}})(b\tau_{\text{out}} - 2a) - (N^2/V_{\text{in}})(b\tau_{\text{in}} - 2a)$. For the liquefaction setup, $\tau_{\text{out}} = \tau_{\text{in}}$, and $\tau_{\text{liq}} = \tau_b$ is the boiling point of the liquid. To first order in $a$ and $b$, we can use the ideal gas equation to write the volumes in terms of the pressures and we get $H_{\text{out}} - H_{\text{in}} = N(b - 2a/\tau_{\text{out}})(p_{\text{out}} - p_{\text{in}}) = Nb(1 - 2a/b\tau_{\text{out}})(p_{\text{out}} - p_{\text{in}})$.

To select the coefficients $a$ and $b$, we set the molar volume of Helium $V = 32cc = 2Nb$ and the inversion temperature $\tau = 51k_B = 2a/b$. The latent heat is $\Delta H = 0.082kJ/mol$, $T_b = 4.18K$, $T_{\text{in}} = 15K$, $p_{\text{out}}$ is the atmospheric pressure. This gives us the estimate (writing all energies in $J$, and pressures in atm):

$$\lambda = \frac{Nb(1 - 2a/b\tau_{\text{out}})(p_{\text{out}} - p_{\text{in}})}{\Delta H + (5/2)N(\tau_{\text{in}} - \tau_{\text{liq}})} = \frac{(32/2) \times 10^{-6}(1 - 51/15)(p_{\text{atm}} - p_{\text{in}})10^5}{(82 + 2.5(15 - 4.18)1.38 \times 10^{-23} \times 6 \times 10^{23})} \approx 0.013(p_{\text{in}} - 1).$$

which is approximately the values in the figure.

**Problem 5.**

(a) The fridge is ideal, so there is no loss of entropy which means that, for a cycle with change in temperature $dT$, the high temperature reservoir fixed at $T_0$, and the low temperature reservoir (the gas) at variable $T$, $dQ_h/T_h = dQ_l/T_l$. The work done for the above cycle is $dW = dQ_h - dQ_l = (T_h/T - 1)dQ_l = -(T_h/T - 1)C_p dT$ for the cooling the ideal gas, where $C_p = (5/2)RT$. Integrating this from $T_0$ to $T_b$ gives $W_1 = -(5/2)R(T_0 - T_b) + (5/2)R\log(T_0/T_b)$. Once the gas has reached boiling point, it remains at a fixed temperature and the heat removed is the latent heat $\Delta H$. The work done for this part is $W_2 = Q_h - Q_l = \Delta H(T_0/T_b - 1)$. Adding all this up gives $W = (5/2)RT_0(\log(T_0/T_b) - (T_0 - T_b)/T_0) + \Delta H(T_0 - T_b)/T_b$.

(b) We have $T_0 = 300K$, $T_b = 4.18K$, $\Delta H = 82J/mol$. Plugging in the numbers gives us $W = 0.228kWh/l$. 
Problem 6. The heat load is 0.1W which is the amount of heat being extracted per second. This is equal to $\Delta H (dN/dt)$ where $\Delta H = 82 J/mol$ is the latent heat of vaporization, and $(dN/dt)$ is the number of moles being extracted per second. The extraction is happening at room temperature, and so $dN/dt = (P/RT_{\text{room}})dV/dt = (P/P_{\text{room}})(1/24)(100) \text{mol/s}$. This gives us $0.1 = 82(P/P_{\text{room}})(1/24)(100) \Rightarrow P = 0.22 \text{torr}$. This is the vapor pressure of the Helium gas, and we read off the temperature to be $T_{\text{min}} \approx 1 \text{K}$. For the smaller heat load of $10^{-3} W$ and a faster pump speed of $10^3 l/s$, we have $P = 2.2 \times 10^{-4} \text{torr}$, and we read off the temperature to be $T_{\text{min}} \approx 0.6 \text{K}$.

Problem 7. A magnetic field of $B_1 = 100 kG$ corresponds to a temperature of $7 K$ (the magnetic moment of the ion is a Bohr magneton). The maximum cooling will be achieved when there is no external magnetic field at the end. The residual magnetic field is then $B_2 = 100 G$ which corresponds to a temperature of $7 mK$. The change in entropy of the spin system is then $N(\tanh^2(7/T_1) - \tanh^2(7/1000T_2))$. The Debye temperature of the lattice is $100 K$. For temperatures much less than $100 K$, the entropy of the lattice is $\sigma_l = N(4\pi^4/5\theta^3)T^3 \approx 8 \times 10^{-5} NT^3$ and the change in entropy for the lattice is $8 \times 10^{-5} N(T_3 - T_2)$. For temperatures much less than $100 K$, the entropy of the lattice overwhelms the spin entropy. Let us say the temperatures are in the range much below the lattice temperature and above the spin temperatures. Then the change in spin entropy can be approximated by $N(7/T_1)^2 - N(7/1000T_2)^2$. For significant cooling ($T_2 = 0.1T_1$), the change in lattice entropy is approximately $N8 \times 10^{-5}T_1^3$, and the change in spin entropy is approximately $N(7/T_1)^2$. Equating the two gives the temperature $T \approx 14 K$ which is consistent with our earlier assumptions. (We can be slightly more precise and use the exact expression for the spin entropies, and find the answer numerically - the answer shifts down a little bit, but we will probably use a slightly lower temperature in an experiment anyway).