Physics 301 Problem Set 3 Solutions

Problem 1.
(a) The momentum flux density is the momentum carried across an imaginary surface of unit area per unit time per unit frequency. Let’s say the normal to the surface is in the $z$ direction. Then, there is no $x$ and $y$ component of momentum carried across this surface. We want to calculate then, the $z$-momentum flux density.
The number of photons with momentum $\mathbf{p} \equiv (p, \theta, \phi) = (\hbar \omega/c, \theta, \phi)$ going through the hole in unit time in the range $\omega$ to $\omega + d\omega$ is the same as the number of photons carrying the above momentum in a cylinder of length $c$, and base area $A$ at angle $(\theta, \phi)$ to the normal from the hole. The $z$-momentum carried by a photon of momentum $\mathbf{p}$ is $\hbar \omega \cos \theta/c$. The $z$-momentum flux carried by these photons is thus:

$$p_z(\omega)d\omega d\Omega = \frac{\hbar \omega}{c} \cos \theta (c \cos \theta) s(\omega) n(\omega) d\omega \frac{d\Omega}{4\pi}. \quad (1)$$

where
1. $(c \cos \theta)$ is the volume of the cylinder,
2. $s(\omega) = (\exp(\hbar \omega/\tau) - 1)^{-1}$ is the average number of photons in the mode $\omega$,
3. $n(\omega)d\omega = \omega^2/\pi^2 c^3$ is the number of modes per unit volume in the range $\omega$ to $\omega + d\omega$ (travelling in all directions); and
4. $d\Omega/4\pi = \sin \theta d\theta d\phi$ is the fraction of the photons that are travelling in the direction of the hole. Integrating over the angles gives the momentum flux density:

$$p_z(\omega)d\omega = \hbar \omega s(\omega) n(\omega) d\omega \frac{1}{4\pi} \int_0^{2\pi} d\phi \int_0^\pi \sin \theta d\theta \cos^2 \theta$$

$$= \hbar \omega s(\omega) n(\omega) d\omega \frac{2\pi}{4\pi} \frac{2}{3} = \frac{1}{3} u(\omega)d\omega. \quad (2)$$

where $u(\omega)$ is the spectral density. The pressure is given by summing over the frequencies (we can either do the integral or look up the total energy which we have seen many times before):

$$p = \int_0^\infty d\omega \frac{1}{3} u(\omega) = \frac{1}{3} u. \quad (3)$$

(b) For photons in a cubical box, the frequencies are given by $\omega = \text{const}/L$; $V = L^3$ which gives:

$$\frac{dV}{V} = 3 \frac{dL}{L} = -3 \frac{d\omega}{\omega}. \quad (4)$$
For constant entropy processes, the occupation numbers $\langle s \rangle = (\exp(\hbar \omega/\tau) - 1)^{-1}$ don't change, and this means that $d(\omega/\tau) = 0$ which gives us the fourth relation $dV/V = -3d\omega/\omega = -3d\tau/\tau$.

The energy of a photon gas is $U = AV\tau^4$ and the entropy is $\sigma = BV\tau^3$. The energy can therefore be expressed as $U = C\sigma\tau$ where $A, B$ and $C$ are constants. The pressure is then given by:

$$p \equiv -\left( \frac{\partial U}{\partial V} \right)_\sigma = -\left( \frac{\partial U}{\partial \tau} \right)_\sigma = -\frac{C\sigma}{(3V)} = \frac{U}{3V} = \frac{u}{3}. \quad (5)$$

**Problem 2.**

The average number of photons per mode is

$$\langle n(\omega) \rangle = \frac{1}{\exp(h\omega/\tau) - 1} \quad (6)$$

Using the same density of states as to count the total energy, the total number of photons are

$$N = \int_0^\infty \frac{d\omega}{\pi^2 c^3} \langle n(\omega) \rangle = \frac{V}{\pi^2 c^3} \int_0^\infty \frac{d\omega}{\exp(h\omega/\tau) - 1} \omega^2 \exp(h\omega/\tau) - 1$$

$$= \frac{V\tau^3}{\pi^2 c^3 \hbar^3} \int_0^\infty dx \frac{x^2}{e^x - 1} = \frac{V\tau^3}{\pi^2 c^3 \hbar^3} I \frac{V\tau^3}{\pi^2 c^3 \hbar^3} 2\zeta(3) \approx 2.404 \frac{V\tau^3}{\pi^2 c^3 \hbar^3}. \quad (7)$$

The entropy of the photon gas is

$$\sigma = \frac{4\pi^2}{45\hbar^3 c^3} V\tau^3 = \frac{4\pi^4}{45 \times 2.404} N \approx 3.6N. \quad (8)$$

**Problem 3.**

Let the area of the small hole be $dA$. We want to count the number of photons with momentum $p \equiv (p, \theta, \phi) = (h\omega/c, \theta, \phi)$ going through the hole in unit time in the range $\omega$ to $\omega + d\omega$. This is the same as the number of photons carrying the above momentum in a cylinder of length $c$, and base area $A$ at angle $(\theta, \phi)$ to the normal from the hole. (Drawing such a cylinder would make this statement clear). This number is $f(\omega)dAd\Omega d\omega = (dA \cos \theta) s(\omega) n(\omega) d\omega d\Omega/4\pi$ (f denotes the number flux density) where

1. $(dA \cos \theta)$ is the volume of the cylinder,
2. $s(\omega) = (\exp(h\omega/\tau) - 1)^{-1}$ is the average number of photons in the mode $\omega$,
3. $n(\omega)d\omega = \omega^2/\pi^2 c^3$ is the number of modes per unit volume in the range $\omega$ to $\omega + d\omega$ (travelling in all directions); and
4. \( d\Omega/4\pi = \sin \theta d\theta d\phi \) is the fraction of the photons that are travelling in the direction of the hole.

The number of photons in the above frequency range going through the hole per unit area per unit time is given by an integration over the angles, and since each photon carries energy \( \hbar \omega \), the energy flux density is:

\[
E_{\text{hole}}(\omega) d\omega = c \hbar s(\omega)n(\omega) d\omega \frac{1}{4\pi} \int_{0}^{2\pi} d\phi \int_{0}^{1} (d\cos \theta) \cos \theta.
\]

\[= \frac{1}{4} cs(\omega)n(\omega)d\omega = \frac{1}{4} cu(\omega)d\omega. \tag{9}\]

where \( u(\omega) \) is the spectral density.

The total energy going through the hole is therefore given by summing the above density over all the frequencies, and the answer is (this is basically the same integral as in problem 1, and in the calculation of total energy):

\[
F = \frac{1}{4} cu = \frac{\pi^2}{60\hbar^3 c^2} \tau^4. \tag{10}\]

**Problem 4.**

The gravitational self energy of the sun is (by dimensional arguments) \( E = -GM^2/R \) where the mass and the radius are those of the sun. (There is also a number in front of order 1). By the virial theorem, the average kinetic energy is \( K = -E/2 = GM^2/2R \). Assuming the particles in the sun to be an ideal gas, \( K = 3Nk_B T/2 \) which gives the estimate of the temperature as \( T \approx GM^2/kN = 3 \times 10^7 K \) for the numbers given.

**Problem 5.**

(a) For a given mode of the photon gas with frequency \( \omega \), the allowed states are labelled by an integer \( k \) which is the occupancy (number of photons) of the mode. The energies are \( k\hbar \omega \). The partition function is therefore

\[
Z(\omega) = \sum_{0}^{\infty} \exp(-k\hbar \omega) = (1 - \exp(-\hbar \omega/\tau))^{-1} \tag{11}\]

Since the modes are independent of each other, the full partition sum is the product of the one-mode partition sums over all the modes:

\[
Z = \prod_{n}(1 - \exp(-\hbar \omega_n/\tau))^{-1} \tag{12}\]
(b) The free energy is
\[ F = -\tau \log Z = -\tau \log \prod_n (1 - \exp(-\hbar \omega_n/\tau))^{-1} = \tau \sum_n \log (1 - \exp(-\hbar \omega_n/\tau)). \] (13)

The frequencies are \( \omega_n = n\pi c/L \) where \( L \) is the size of the box, and we can approximate the sum by an integral by using the usual density of states:
\[ F \approx \int_0^\infty \tau \log(1 - \exp(-\hbar \omega_n/\tau)) \frac{V}{\pi^2 c^3} \omega^2 d\omega \]
\[ = \frac{\tau V}{\pi^2 c^3 h^3} \left[ \frac{1}{3} x^3 \log(1 - e^{-x}) \right]_0^\infty - \int_0^\infty dx \frac{1}{3} \frac{x^3 e^{-x}}{(1 - e^{-x})} \]
\[ = \frac{\tau^4 V}{\pi^2 c^3 h^3} \left[ 0 - \frac{1}{3} \frac{\pi^4}{15} \right] = -\frac{\tau^4 V \pi^2}{45 c^3 h^3} \] (14)

**Problem 6.**

The middle plane absorbs radiation from both the planes, and radiates back on both the sides. The power absorbed by unit area of the middle plane is \( \sigma_B T_u^4 + \sigma T_l^4 \), and the power radiated by the same area is \( 2\sigma_B T_m^4 \). Equilibrium would mean that these two are equal which gives
\[ T^4 = \frac{1}{2}(T_u^4 + T_l^4). \] (15)

The radiation flux between the upper and middle plane is
\[ J = \sigma_B T_u^4 - \sigma_B T_m^4 = \sigma_B T_u^4 - \frac{1}{2} \sigma_B (T_u^4 + T_l^4) = \frac{1}{2} \sigma_B (T_u^4 - T_l^4) = \frac{1}{2} J_{\text{initial}} \] (16)
which is also the radiation between the middle and lower side.

**Problem 7.**

The total energy of a solid due to the elastic waves in the solid is
\[ U = \frac{3V \tau^4}{2\pi^2 h^3 v^3} \int_0^{x_D} dx \frac{x^3}{e^x - 1}. \] (17)
where \( v \) is the velocity of sound in the solid, and \( x_D = \theta/\tau \) where \( \theta = (\hbar v/k_B)(6\pi^2 N/V)^{1/3} \) the Debye temperature is related to the maximum frequency allowed. In the limit \( T \gg \theta \), \( x_D \ll 1 \), \( e^x - 1 \approx x \); and the integral can be approximated as:
\[ U \approx \frac{3V \tau^4}{2\pi^2 h^3 v^3} \int_0^{x_D} dx x^2 = \frac{3V \tau^4}{2\pi^2 h^3 v^3} \frac{x_D^3}{3} = 3Nk_B T. \] (18)
For $T$ only moderately larger than $\theta$, we get a better approximation by expanding the integral further in powers of $x_D$, which means expanding the function $x^3(e^x-1)^{-1}$ further in powers of $x$. Expanding to two orders would give $U = 3NkT(1 + a_1(\theta/T) + a_2(\theta/T)^2 + \ldots)$. The heat capacity is thus $C_v = 3Nk(1 - a_2(\theta/T)^2 + \ldots)$. The first non-zero correction is thus $a_2$ which we shall calculate. To this order,

$$
(e^x - 1)^{-1} \approx (x + \frac{x^2}{2} + \frac{x^3}{6})^{-1} = \frac{x}{x(1 + \frac{x}{2} + \frac{x^2}{6})} = \frac{1}{x}(1 - \frac{x^2}{2} - \frac{x^2}{6} + \frac{x^2}{4}) = \frac{1}{x}(1 - \frac{x}{2} + \frac{x^2}{12}).
$$

This gives

$$
\int_0^{x_D} dx \frac{x^3}{e^x - 1} \approx \int_0^{x_D} dx \left(1 - \frac{x}{2} + \frac{x^2}{12}\right) = \frac{x_D^3}{3} - \frac{x_D^4}{8} + \frac{x_D^5}{60}
$$

which gives the value $a_2 = 1/20$. Looking up table 4.2, we find $C_v(0) = 24.93$ for $\theta/T = 0$ and $C_v(1) = 23.74$ for $\theta/T = 1$. If we ignore higher approximations, this gives $a_2 = (C_v(1) - C_v(0))/C_v(1) \approx 1.19/23.74 \approx 0.048$ which is off from $1/20 = 0.05$ by 4%.

**Problem 8.**

(a) If we have only longitudinal sound waves, the density of states is less than the usual one by a factor of 3: $n(\omega) = (V/2\pi^2v^3)\omega^2$. The number of modes of the solid remain $3N$. The Debye temperature in this case is given by

$$
3N = \int_0^{\omega_D} n(\omega)d\omega = \frac{V}{2\pi^2v^3}\omega_D^3
$$

$$
\Rightarrow \omega_D = \left(\frac{18\pi^2Nv^3}{V}\right)^{1/3} = (18\pi^2\rho)^{1/3}v.
$$

The density is $0.145g/cc = 0.145 \times 6.023 \times 10^{23}/4 = 2.18 \times 10^{22}/cc$. This gives $\theta_D = h\omega_D/k = (18\pi^2\rho)^{1/3}(hv/k) \approx 28.5K$.

(b) The total energy is

$$
U = \frac{V}{2\pi^2v^3} \int_0^{\omega_D} \frac{\omega^2h\omega}{e^{h\omega/\tau} - 1}d\omega = \frac{V}{2\pi^2h^3v^3}\tau^4 \int_0^{x_D} \frac{x^3}{e^x - 1}
$$

$$
\approx \frac{V}{2\pi^2h^3v^3}\tau^4 \int_0^{\infty} \frac{x^3}{e^x - 1} = \frac{V}{2\pi^2h^3v^3}\tau^4 \frac{\tau^4}{15}
$$

$$
= \frac{V\pi^2}{30h^3v^3}\tau^4 = \frac{3\pi^4v^2\rho}{5k^3\theta_D^3}\tau^4 = \frac{3\pi^4Nk}{5\theta_D^3}\tau^4
$$

$$
\Rightarrow C_v = \frac{12}{5}N\pi^4k(\frac{T}{\theta_D})^3 \approx 1.39 \times 10^{-25}NT^3J/K.
$$
\[ N = N_A \text{ is 4g of He, which means that the specific heat per gram is } C_v \approx (1.39 \times 10^{-25} \times 6.02 \times 10^{23}/4) T^3 \approx 0.021 T^3 J/gK. \]