When you are told to begin, check that this examination booklet contains all the numbered pages from 2 through 16.

The exam contains six problems that count equally although some problems may be harder than others.

Do not panic or be discouraged if you cannot do every problem; there are both easy and hard parts in this exam. Keep moving and finish as much as you can!

Read each problem carefully. You must show your work—the grade you get depends on how well I can understand your solution even when you write down the correct answer. Please BOX your answers.

DO ALL THE WORK YOU WANT GRADED IN THIS EXAMINATION BOOKLET!

Rewrite and sign the Honor Pledge: I pledge my honor that I have not violated the Honor Code during this examination.

Signature
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Problem 1. Whiplash. A driver of a minivan is proceeding at \( v_{i1} = 30 \) m/s when he spots another minivan only \( D = 100 \) m ahead. He immediately applies the brakes and although his minivan has an anti-lock braking system, he’s traveling on black ice and the best the ABS can do is slow him down with the equivalent of a uniform acceleration of \( a = -1 \) m/s\(^2\). The second (unsuspecting) minivan proceeds at a constant velocity of \( v_{i2} = 10 \) m/s. Take \( t = 0 \) at the instant the brakes are applied and also take the positions of the two minivans at this instant to be \( x_{1}(0) = 0 \) and \( x_{2}(0) = D = 100 \) m.

(a) What are the positions of the minivans as a function of time after \( t = 0 \)? That is, what are \( x_{1}(t) \) and \( x_{2}(t) \)? For now, ignore the impending collision. (4 points)

\[ x_{1}(t) = v_{i1}t + \frac{1}{2}at^2 = \frac{30t - \frac{1}{2}t^2}{2}, \]
\[ x_{2}(t) = D + v_{i2}t = 100 + 10t. \]

(b) Where and when do the minivans collide? (That is, what are \( x \) and \( t \) when they collide?) (6 points).

The collision occurs when \( x_{1}(t) = x_{2}(t) \) for the first time. We set the two equal and after rearranging terms come up with the following equation

\[ -\frac{1}{2}at^2 - (v_{i1} - v_{i2})t + D = 0, \]

This can be solved to give

\[ t_{col} = \frac{(v_{i1} - v_{i2}) \pm \sqrt{(v_{i1} - v_{i2})^2 + 2aD}}{-a}, \]

We want the earlier time so we use the negative sign. Plugging in numbers and remembering that \( a \) is negative, we have \( t_{col} = 5.86 \) s. We get the position by plugging this time into the expression for \( x_{11}(t) \) or \( x_{2i}(t) \). We use the latter since it’s simpler. We find \( x_{col} = D + v_{i2}t_{col} = 158.6 \) m.
Problem 1. Continued.

(c) What is the speed of minivan 1 when it collides with minivan 2? (3 points).

\[ v_1 = v_{i1} + at_{\text{col}} = 30 - 1 \cdot 5.86 = 24.14 \, \text{m/s}. \]

End Solution

(d) The two minivans have a completely inelastic collision. Both minivans have the same mass. What is their speed just after the collision? (4 points)

Conservation of momentum tells us that \[ m_1v_1 + m_2v_{i2} = (m_1 + m_2)v_f. \] Since the masses are the same, \[ v_f = \frac{(v_1 + v_{i2})}{2} = \frac{(24.14 + 10)}{2} = 17.07 \, \text{m/s}. \]

End Solution

(e) What was the coefficient of friction for minivan 1 while it was braking in the unsuccessful attempt to avoid the collision? (3 points)

The friction force is the only horizontal force and, assuming the ABS is working, it’s a static friction and it’s the maximum value of the static friction, in other words, \( f = -\mu_s N \), where \( N = mg \) is the normal force. The friction force produces the given acceleration, \( -\mu_s mg = ma \) or \( \mu_s = -a/g = 1/9.8 = 0.102 \).

End Solution
Problem 2. Pulley and Disk. A pulley—a uniform disk of radius $r_1$ and mass $m_1$—is suspended from the ceiling. It is free to turn about a frictionless bearing through its center. A massless string is wrapped around the pulley and then wrapped around another uniform disk of radius $r_2$ and mass $m_2$. The second disk is positioned below the pulley, so its axis is parallel to the axis of the pulley and the string connecting the disk and pulley is vertical. The second disk is then released and falls straight down, unwrapping string from both the disk and pulley as it falls. Take counterclockwise rotations of the pulley and disk to be positive. Take downward motions (in the $y$ direction indicated on the figure) of the disk to be positive. Assume the string doesn’t stretch and that it doesn’t slip on the pulley nor the disk.

(a) Draw free body diagrams showing all the forces on the pulley and on the disk.

(4 points)

Solution

The forces are the weights acting down through the centers of mass and the tensions acting vertically and tangentially and the force $F$ suspending the pulley from the ceiling and acting through the center of the pulley.

End Solution

(b) Obtain the three equations of motion (that is equations for the angular accelerations of the pulley and the disk and the vertical acceleration of the disk). Express your answers for $\alpha_1$, $\alpha_2$ and $a$ in terms of any or all of $m_1$, $m_2$, $r_1$, $r_2$, $g$ and $T$, the tension in the string. (6 points)

Solution

$\tau = I\alpha$ for the pulley: $Tr_1 = \frac{1}{2} m_1 r_1^2 \alpha_1$;

$\tau = I\alpha$ for the disk: $Tr_2 = \frac{1}{2} m_2 r_2^2 \alpha_2$;

$F = ma$ for the disk: $m_2 g - T = m_2 a$.

End Solution
Problem 2. Continued.

(c) Suppose the pulley rotates through an angle $\theta_1$ (radians). What length of string unwraps from the pulley? If the disk rotates by an angle $\theta_2$ what length of string unwraps from the disk? (3 points)

Solution

Length from pulley is $L_1 = r_1 \theta_1$. Length from disk is $L_2 = r_2 \theta_2$.

End Solution

(d) What is the relation between the linear acceleration of the disk and the angular accelerations of the disk and pulley? (3 points)

Solution

From the previous part, the distance the disk falls when the pulley turns an angle $\theta_1$ and the disk turns an angle $\theta_2$ is $y = L_1 + L_2 = r_1 \theta_1 + r_2 \theta_2$. Differentiating once with respect to time gives a relation between the velocity of the disk and the angular velocities, $v = r_1 \omega_1 + r_2 \omega_2$ and differentiating again gives the desired relation $a = r_1 \alpha_1 + r_2 \alpha_2$.

End Solution

(e) What is the vertical acceleration of the disk? Express your answer in terms of any or all of $m_1, m_2, r_1, r_2$, and $g$. (4 points)

Solution

Starting from the equations of motion for the angular accelerations, we divide through by $mr/2$ to get $2T/m_1 = r_1 \alpha_1$ and $2T/m_2 = r_2 \alpha_2$. Adding and rearranging gives $T = (1/2)[m_1 m_2/(m_1 + m_2)]a$. Putting this in the equation of motion for the acceleration and rearranging gives

$$a = \frac{2m_1 + 2m_2}{3m_1 + 2m_2} g.$$

End Solution
Problem 3. The Crane. A crane is made from a uniform rigid boom of length $L = 30$ m and weight $w = 400$ N. The boom is supported by a frictionless hinge at its lower end. Its upper end is attached to a support cable which is fastened behind the boom at the same horizontal level as the hinge. Adjusting the length of the support cable adjusts the angle of the boom. In a particular application, the crane is used to support a crate with weight $W = 2,000$ N. The boom makes an angle $\theta = 45^\circ$ with the horizontal and the support cable makes an angle $\phi = 30^\circ$ with the horizontal.

(a) Draw a free body diagram for the boom showing all the forces acting on the boom. Be sure to indicate the point of application of the force and be sure to label all the forces. (5 points)

Solution

The forces include the weight, $w$, of the boom acting down through the center of the boom (since the boom is uniform), the weight, $W$, of the crate acting down at the upper end of the boom, the tension, $T$, in the support cable acting at the upper end of the boom along the cable, and the horizontal, $F_x$, and vertical, $F_y$ components of the force exerted by the hinge, acting at the lower end of the boom.

End Solution

(b) Determine the tension in the support cable. (6 points)

Solution

If we balance torques about the hinge, the force from the hinge has zero lever arm and drops out. The only unknown in the torque balance equation is the tension in the support cable. The lever arm for $w$ is $(L/2) \cos \theta$ (perpendicular distance from origin—hinge—to line of action of $w$). The lever arm for $W$ is $L \cos \theta$. The lever arm for $T$ is $L \sin(\theta - \phi)$. Note that $\theta$ is an exterior angle of the triangle formed by the boom, the support cable, and the horizontal, so it is the sum of $\phi$ and the angle at the upper end of the boom. The torques from the weights are negative; the torque from the tension is positive, so $TL \sin(\theta - \phi) - w(L/2) \cos \theta - WL \cos \theta = 0$. Solving for $T$,

$$T = \frac{\cos \theta}{\sin(\theta - \phi)} (W + w/2) = \boxed{6011 \text{ N}}.$$ 

End Solution
Problem 3. Continued.

(c) Determine the horizontal and vertical components of the force exerted by the hinge on the boom. (4 points)

Solution

We balance the horizontal and vertical components of all the forces acting on the boom. For the horizontal, we have $F_x - T \cos \phi = 0$ which gives $F_x = T \cos \phi = 5205$ N. For the vertical we have $F_y - w - W - T \sin \phi = 0$, so $F_y = w + W + T \sin \phi = 5405$ N.

End Solution

(d) As mentioned, the support cable can be lengthened or shortened to change the angle of the boom. Suppose the boom is lowered towards the horizontal by lengthening the cable. Explain why you expect the crane to eventually fail—the cable snaps or pulls loose from its fastening, the boom buckles, or the hinge breaks—as the boom nears the horizontal. (5 points)

Solution

The lever arm for the cable decreases to zero as the boom becomes horizontal. Therefore the tension in the cable must increase to $\infty$ as the boom becomes horizontal. This means the horizontal force from the hinge also increases to $\infty$ as does the compression of the boom. Something has to give. Either the cable snaps or breaks away from its fastening, the boom buckles, or the hinge breaks!

End Solution
Problem 4. Lunar Orbits. An astronaut is in a spacecraft in a circular orbit about the Moon. The mass of the spacecraft and contents is $m = 10,000$ kg, the radius of the orbit is $R = 1850$ km, the period of the orbit is $T = 120$ min $= 7200$ s, the radius of the moon (assumed spherical) is $r = 1740$ km, and Newton’s constant of gravitation is $G = 6.67 \times 10^{-11}$ m$^3$/kg s$^2$.

(a) From the data given, calculate the mass of the Moon. (4 points)

Solution

\[
\frac{GMm}{R^2} = \frac{mv^2}{R} = 4\pi^2mR/T^2, \quad \text{so} \quad M = \frac{4\pi^2R^3}{GT^2} = 7.23 \times 10^{22} \text{ kg}.
\]

Close to the tabulated value of the lunar mass: $7.35 \times 10^{22}$ kg.

End Solution

(b) What are the spacecraft’s speed (relative to the moon), its angular momentum, and its total mechanical energy? (5 points)

Solution

\[ v = 2\pi R/T = 1614 \text{m/s}, \quad L = mvR = 2.99 \times 10^{13} \text{kg m}^2/\text{s}, \quad E = -GMm/R + \frac{mv^2}{2} = -1.30 \times 10^{13} \text{J}. \]

End Solution

The astronaut decides that she wants some closeup pictures of the Moon. To do this, at point A, she ejects an unpowered probe in a direction—relative to the spacecraft—directly opposite that of the spacecraft. The probe then follows an elliptical orbit (shown by the dashed line in the figure on the previous page) and just grazes the surface of the Moon at point B. Compared to the spacecraft, the mass, \( m_p = 10 \text{kg} \), of the probe is negligible. Note that the orbit of the probe and that of the spacecraft are both in the counterclockwise direction in the figure.

(c) What is the speed with which the probe must be ejected (relative to the spacecraft) to put the probe in the desired elliptical orbit? (6 points)

Solution

We have an elliptical orbit in which we know the perilune, point B, \( r_p = r \), and the apolune, point A, \( r_a = R \). We want to know the velocity at apolune. We use conservation of angular momentum and conservation of energy. \( m_pr_av_a = m_pr_pv_p \) and \(-GMm_p/r_a + m_pv_p^2/2 = -GMm_p/r_p + m_pv_p^2/2\). Solve the angular momentum equation for \( v_p \) and plug into the energy equation. The mass of the probe drops out and the result for \( v_a \) is

\[ v_a = \sqrt{\frac{2GMr_p}{(r_a + r_p)r_p}} = 1590 \text{m/s}. \]

But this is the speed relative to the Moon. We must subtract the speed of the spacecraft to get the relative speed, \( v_r = -24 \text{m/s} \), where the minus sign indicates the probe is ejected backwards.

End Solution

(d) What impulse must be given to the probe in order to produce the speed calculated in part (c)? (3 points)

Solution

The impulse is just the change in momentum which is just the mass times the change in velocity, so \( I = m_p v_r = -240 \, \text{kg m/s} \) and again the minus sign indicates the impulse is directed backwards.

End Solution

(e) Assuming the probe doesn’t crash, it will return to point A, as will the spacecraft. Which returns first, the spacecraft or the probe? (2 points)

Solution

The probe wins the race. Its orbit has a smaller semi-major axis so it has a shorter period (Kepler’s third law) which means it completes the orbit (A to A) in a shorter time. Even though the spacecraft is travelling faster at A, it has farther to go. Also the probe will be going faster at perilune.

End Solution
Problem 5. Torsional Oscillator. A uniform disk of radius $R$ and mass $M$ is attached to a rod along its symmetry axis. (See figure labeled “Oblique View.”) The rod is vertical and rigidly attached to supports (not shown) at either end. The mass of the rod is negligibly small. The only possible motion of the disk is rotation about its axis. This twists the rod which provides a restoring torque $\tau = -\kappa \theta$ when the disk is rotated from equilibrium by the angle $\theta$. The torsion constant, $\kappa$, of the rod has units Newton · meter/radian.

(a) What is the frequency, $\omega$, of torsional (twisting) oscillations? (4 points)

Solution

The equation of motion is $\tau = I \ddot{\theta}$, $\tau = -\kappa \theta$, and $I = MR^2/2$. Altogether,

$$\frac{d^2 \theta}{dt^2} + \frac{2\kappa}{MR^2} \theta = 0,$$

and we can read off the frequency: $\omega = \sqrt{2\kappa/\kappa}$.

End Solution

A bullet of mass $m$ is fired horizontally at the disk with velocity $v$. The bullet embeds itself in the rim of the disk as shown in “Top View” above. The path of the bullet is such that it would have passed the rotation axis at distance $b$ if it had not been stopped by the disk. The disk, initially at rest, is set into torsional oscillations by the impact of the bullet. Assume that the mass of the bullet is sufficiently small compared to the mass of the disk that the bullet’s contribution to the moment of inertia of the system may be neglected.
Problem 5. Continued.

(b) What is the angular velocity, $\Omega$, of the disk immediately after impact? (4 points)

Solution

The initial angular momentum about the rotation axis is carried by the bullet and is $mv_b$ ($b$ is the lever arm). The final angular momentum is due to the rotation of the disk and is $I\Omega = (MR^2/2)\Omega$. Since there are no external torques about the rotation axis, the angular momentum of the system is conserved and $\Omega = \frac{2mv_b/MR^2}{2}$.

End Solution

(c) Take $t = 0$ to be the instant when the bullet impacts the disk and take the positive direction of $\theta$ to be counterclockwise in the figure. Obtain an expression for $\theta(t)$ for times after the impact of the bullet ($t = 0$). Only knowns should appear in your answer. These include, $m, M, R, b, v,$ and $\kappa$. Of course $t$ is allowed! (6 points)

Solution

The general solution is $\theta(t) = A \cos(\omega t + \delta)$. We’ve already figured out $\omega$. We need to figure out $A$ and $\delta$. What we know is that at $t = 0$ the displacement of the disk is 0 and its angular velocity is the $\Omega$ we calculated in part (b). This gives two equations: $0 = A \cos \delta$ and $\Omega = -\omega A \sin \delta$. To satisfy the first without making $A = 0$, we must have $\delta = \pi/2$ or $\delta = -\pi/2$. We can pick the sign by looking at the second equation. The angular velocity is positive at $t = 0$, so we want $\delta = -\pi/2$. Then the second equation becomes $\Omega = \omega A$ and $A = \Omega/\omega = \frac{2mv_b}{\sqrt{2\kappa MR^2}}$. Altogether,

$$\theta(t) = \frac{2mv_b}{\sqrt{2\kappa MR^2}} \cos \left( \sqrt{\frac{2\kappa}{MR^2}} t - \frac{\pi}{2} \right) = \frac{2mv_b}{\sqrt{2\kappa MR^2}} \sin \left( \sqrt{\frac{2\kappa}{MR^2}} t \right).$$

End Solution
Problem 5. Continued.

In the following, circle the correct answers. There should be 6 circles. No explanation is required. One point each.

(d) If the initial speed of the bullet is doubled, the amplitude of the oscillations decreases stays the same increases
and the period of the oscillations decreases stays the same increases.

Solution

The period is determined by the moment of inertia of the disk and the torsion constant of the rod. These are unchanged by changing the mass of the bullet. Doubling the mass of the bullet doubles the post collision angular velocity of the disk, so the amplitude doubles.

End Solution

(e) If the mass of the disk is doubled, the amplitude of the oscillations decreases stays the same increases
and the period of the oscillations decreases stays the same increases.

Solution

Doubling the mass of the disk increases the moment of inertia and increases the period (by $\sqrt{2}$). The post collision angular velocity of the disk is halved. The post collision energy is doubled (because $I$ doubles), but halved and halved again (because $\Omega$ halves). The energy is $(1/2)\kappa A^2$ so the amplitude must decrease by $\sqrt{2}$.

End Solution

(f) If the length of the rod is doubled, the amplitude of the oscillations decreases stays the same increases
and the period of the oscillations decreases stays the same increases.

Solution

This is the only tricky one. Doubling the length of the rod halves the twist per unit length for a given displacement from equilibrium. Each part of the rod is under half the strain. So the torsion constant, $\kappa$, halves. It’s analogous to connecting two springs in series. The period increases by $\sqrt{2}$. Since the post collision energy of the disk stays the same, the amplitude must increase by $\sqrt{2}$ to make up for the halving of the torsion constant.

End Solution
**Problem 6. Smacked by a Pendulum.** A uniform thin rod of length $a$ and mass $M$ is pivoted at one end on a frictionless bearing. It is pulled to the side so that the other end rises by $2h$ (and the center of mass by $h$) and released from rest. It swings down and at the very bottom of the swing when it’s vertical, the end smacks a (point) mass, $m$, initially at rest on a frictionless, horizontal surface. The rod and the point mass have an elastic collision. Note that linear momentum is NOT conserved during this collision.

(a) What is the angular velocity, $\omega_i$, of the rod about its pivot just before smacking the mass when the rod is vertical? (4 points)

**Solution**

Use conservation of energy. $Mgh = I \omega_i^2/2$, $I = Ma^2/3$, so $\omega_i = \sqrt{6gh/a}$.

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(b) In terms of $\omega_i$, $M$, and $a$, what are the angular momentum of the rod about the pivot and the kinetic energy of the rod just before the collision with the mass? (2 points)

**Solution**

$L_i = I \omega_i = Ma^2 \omega_i/3$, $K_i = I \omega_i^2/2 = Ma^2 \omega_i^2/6$.

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Problem 6. Continued.

(c) Just after the elastic collision, the angular velocity of the rod is $\omega_f$ and the linear velocity of the point mass is $v$. In terms of these variables as well as $M, m,$ and $a$, what are the angular momentum of the system (rod plus point mass) about the pivot and the kinetic energy of the system just after the collision? (6 points)

Solution

\[ L_f = I \omega_f + mva = Ma^2 \omega_f / 3 + mva, \quad K_f = I \omega_f^2 / 2 + mv^2 / 2 = Ma^2 \omega_f^2 / 6 + mv^2 / 2. \]

End Solution

(d) Determine the velocity of the puck just after the collision in terms of $\omega_i, m, M$ and $a$. (4 points)

Solution

We use conservation of angular momentum and conservation of energy.

\[ \frac{1}{3} Ma^2 \omega_i = \frac{1}{3} Ma^2 \omega_f + mva, \quad \frac{1}{6} Ma^2 \omega_i^2 = \frac{1}{6} Ma^2 \omega_f^2 + \frac{1}{2} mv^2. \]

Divide through by $Ma^2 / 3$ and $Ma^2 / 6$ and move the $v$ term to the left hand side.

\[ \omega_i - 3 \frac{m}{M} \frac{v}{a} = \omega_f, \quad \omega_i^2 - 3 \frac{m}{M} \frac{v^2}{a^2} = \omega_f^2. \]

Square the angular momentum equation and subtract from the energy equation to eliminate $\omega_f$. The result is

\[ 6 \frac{m}{M} \frac{v}{a} \omega_i - \left( 3 \frac{m}{M} + 9 \frac{m^2}{M^2} \right) \frac{v^2}{a^2} = 0, \]

which has two solutions for $v$. One solution is $v = 0$ which corresponds to no collision, not the solution we want. The other solution is

\[ v = \frac{2 \omega_i a}{1 + 3m/M}. \]

End Solution
Problem 6. Continued.

(e) As noted earlier, linear momentum in the horizontal direction is not conserved during the collision. What is the external horizontal force that acts to change the horizontal linear momentum during the collision? (4 points)

Solution

The bearing at the pivot must supply an external horizontal impulse during the collision to keep the pivot end of the rod at rest. If the collision took place at the center of percussion, rather than the end, of the rod, the impulse from the bearing would be zero and linear momentum would be conserved. The center of percussion is $2a/3$ from the pivot end of the rod. (Working this out is left as an exercise!)

End Solution
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