1. Spinor rotations. Somewhat based on a problem in Schwabl.

(a) Suppose $\boldsymbol{n}$ is a unit vector. We are interested in $\boldsymbol{n}\cdot\boldsymbol{\sigma}$. Show that $(\boldsymbol{n}\cdot\boldsymbol{\sigma})^2 = I$, where I is the identity matrix.

Solution

We will use the properties of the Pauli matrices that $\sigma_i^2 = I$ and $\{\sigma_i, \sigma_j\} = 0$, provided $i \neq j$.

$$\begin{aligned}
(\boldsymbol{n}\cdot\boldsymbol{\sigma})^2 &= (n_x\sigma_x + n_y\sigma_y + n_z\sigma_z)(n_x\sigma_x + n_y\sigma_y + n_z\sigma_z) \\
&= n_x^2\sigma_x^2 + n_y^2\sigma_y^2 + n_z^2\sigma_z^2 \\
&\qquad + n_xn_y(\sigma_x\sigma_y + \sigma_y\sigma_x) + n_yn_z(\sigma_y\sigma_z + \sigma_z\sigma_y) + n_zn_x(\sigma_z\sigma_x + \sigma_x\sigma_z) \\
&= (n_x^2 + n_y^2 + n_z^2)I \\
&= I
\end{aligned}$$

End Solution

The unitary operator which generates a rotation by φ about the axis $\boldsymbol{n}$ in spinor space is

$$U = e^{i\varphi\,\boldsymbol{n}\cdot\boldsymbol{S}/\hbar}\,.$$

(b) Show that $U = \cos\varphi/2 + i\boldsymbol{n}\cdot\boldsymbol{\sigma}\sin\varphi/2$.

Solution

First, we note that $\boldsymbol{n}\cdot\boldsymbol{S}/\hbar = \boldsymbol{n}\cdot\boldsymbol{\sigma}/2$. Second, we expand the exponential in a Taylor series and note that since $(\boldsymbol{n}\cdot\boldsymbol{\sigma})^2 = I$, even terms in the series will involve the identity matrix and odd terms will involve $\boldsymbol{n}\cdot\boldsymbol{\sigma}$ to the first power.

$$\begin{aligned}
U = e^{i\varphi\,\boldsymbol{n}\cdot\boldsymbol{S}/\hbar} &= e^{i(\varphi/2)(\boldsymbol{n}\cdot\boldsymbol{\sigma})} \\
&= I\sum_{n=0,2,4,\ldots}\frac{i^n}{n!}(\varphi/2)^n + i(\boldsymbol{n}\cdot\boldsymbol{\sigma})\sum_{n=1,3,5,\ldots}\frac{i^{n-1}}{n!}(\varphi/2)^n \\
&= I\cos\varphi/2 + i(\boldsymbol{n}\cdot\boldsymbol{\sigma})\sin\varphi/2\,,
\end{aligned}$$

End Solution

(c) Show that

$$U\boldsymbol{\sigma}U^\dagger = \boldsymbol{n}(\boldsymbol{n}\cdot\boldsymbol{\sigma}) - \boldsymbol{n}\times(\boldsymbol{n}\times\boldsymbol{\sigma})\cos\varphi + (\boldsymbol{n}\times\boldsymbol{\sigma})\sin\varphi\,.$$

You might find it helpful to remember that a product of two Pauli matrices gives the identity matrix or plus or minus i times the third Pauli matrix according to the formula

$$\sigma_i\sigma_j = \delta_{ij} + i\epsilon_{ijk}\sigma_k\,,$$

where the summation convention is assumed and where we have just used 1 in place of I.

Solution

$$U\boldsymbol{\sigma}U^\dagger = (\cos\varphi/2 + i\boldsymbol{n}\cdot\boldsymbol{\sigma}\sin\varphi/2)\boldsymbol{\sigma}(\cos\varphi/2 - i\boldsymbol{n}\cdot\boldsymbol{\sigma}\sin\varphi/2)$$
$$= \cos^2\varphi/2\,\boldsymbol{\sigma} + i\cos\varphi/2\sin\varphi/2(\boldsymbol{n}\cdot\boldsymbol{\sigma}\,\boldsymbol{\sigma} - \boldsymbol{\sigma}\,\boldsymbol{n}\cdot\boldsymbol{\sigma}) + \sin^2\varphi/2\,\boldsymbol{n}\cdot\boldsymbol{\sigma}\,\boldsymbol{\sigma}\,\boldsymbol{n}\cdot\boldsymbol{\sigma}$$

At this point, we just grind out the $i^{\rm th}$ component of each term. The first term is just $\cos^2\varphi/2\,\sigma_i$. For the second term, we need

$$\begin{aligned}
\boldsymbol{n}\cdot\boldsymbol{\sigma}\,\sigma_i - \sigma_i\,\boldsymbol{n}\cdot\boldsymbol{\sigma} &= n_j\sigma_j\sigma_i - \sigma_i n_j\sigma_j \\
&= n_j[\sigma_j,\sigma_i] \\
&= 2in_j\epsilon_{jik}\sigma_k \\
&= -2i\epsilon_{ijk}n_j\sigma_k \\
&= -2i\,(\boldsymbol{n}\times\boldsymbol{\sigma})_i\ ,
\end{aligned}$$

and the middle term is

$$\sin\varphi\,(\boldsymbol{n}\times\boldsymbol{\sigma})_i\ .$$

For the third term we need

$$\begin{aligned}
\boldsymbol{n}\cdot\boldsymbol{\sigma}\,\sigma_i\,\boldsymbol{n}\cdot\boldsymbol{\sigma} &= n_j\sigma_j\sigma_i n_k\sigma_k \\
&= n_j n_k\sigma_j\sigma_i\sigma_k \\
&= n_j n_k(\delta_{ji} + i\epsilon_{jil}\sigma_l)\sigma_k \\
&= n_i\,\boldsymbol{n}\cdot\boldsymbol{\sigma} + in_jn_k\epsilon_{jil}(\delta_{lk} + i\epsilon_{lkm}\sigma_m) \\
&= n_i\,\boldsymbol{n}\cdot\boldsymbol{\sigma} - n_j\epsilon_{jil}\epsilon_{lkm}n_k\sigma_m \\
&= n_i\,\boldsymbol{n}\cdot\boldsymbol{\sigma} - n_j\epsilon_{jil}(\boldsymbol{n}\times\boldsymbol{\sigma})_l \\
&= n_i\,\boldsymbol{n}\cdot\boldsymbol{\sigma} + \epsilon_{ijl}n_j(\boldsymbol{n}\times\boldsymbol{\sigma})_l \\
&= n_i\,\boldsymbol{n}\cdot\boldsymbol{\sigma} + (\boldsymbol{n}\times(\boldsymbol{n}\times\boldsymbol{\sigma}))_i
\end{aligned}$$

The second term looks like what we want, but we still have more work to do on the first and third terms. We combine the two terms and use the identities $\cos^2\varphi/2 = (1+\cos\varphi)/2$ and $\sin^2\varphi/2 = (1-\cos\varphi)/2$ to get

$$(\boldsymbol{\sigma} + \boldsymbol{n}(\boldsymbol{n}\cdot\boldsymbol{\sigma}) + \boldsymbol{n}\times(\boldsymbol{n}\times\boldsymbol{\sigma}))/2 + \cos\varphi(\boldsymbol{\sigma} - \boldsymbol{n}(\boldsymbol{n}\cdot\boldsymbol{\sigma}) - \boldsymbol{n}\times(\boldsymbol{n}\times\boldsymbol{\sigma}))/2)\ .$$

We now use the vector identity $\boldsymbol{A}\times(\boldsymbol{B}\times\boldsymbol{C}) = \boldsymbol{B}(\boldsymbol{A}\cdot\boldsymbol{C}) - \boldsymbol{C}(\boldsymbol{A}\cdot\boldsymbol{B})$. In the first term above,

$$(\boldsymbol{\sigma} + \boldsymbol{n}(\boldsymbol{n}\cdot\boldsymbol{\sigma}) + \boldsymbol{n}\times(\boldsymbol{n}\times\boldsymbol{\sigma}))/2 = (\boldsymbol{\sigma} + \boldsymbol{n}(\boldsymbol{n}\cdot\boldsymbol{\sigma}) + \boldsymbol{n}(\boldsymbol{n}\cdot\boldsymbol{\sigma}) - \boldsymbol{\sigma}(\boldsymbol{n}\cdot\boldsymbol{n}))/2 = \boldsymbol{n}(\boldsymbol{n}\cdot\boldsymbol{\sigma})\ .$$

In the second term, we have

$$\begin{aligned}\cos\varphi(\boldsymbol{\sigma} - \boldsymbol{n}(\boldsymbol{n}\cdot\boldsymbol{\sigma}) - \boldsymbol{n}\times(\boldsymbol{n}\times\boldsymbol{\sigma}))/2 &= \cos\varphi(\boldsymbol{\sigma} - \boldsymbol{n}(\boldsymbol{n}\cdot\boldsymbol{\sigma}) - \boldsymbol{n}(\boldsymbol{n}\cdot\boldsymbol{\sigma}) + \boldsymbol{\sigma}(\boldsymbol{n}\cdot\boldsymbol{n}))/2 \\ &= \cos\varphi(\boldsymbol{\sigma} - \boldsymbol{n}(\boldsymbol{n}\cdot\boldsymbol{\sigma})) \\ &= -\cos\varphi(\boldsymbol{n}\times(\boldsymbol{n}\times\boldsymbol{\sigma}))\,.\end{aligned}$$

Putting everything together, we get the desired result:

$$U\boldsymbol{\sigma}U^\dagger = \boldsymbol{n}(\boldsymbol{n}\cdot\boldsymbol{\sigma}) - \boldsymbol{n}\times(\boldsymbol{n}\times\boldsymbol{\sigma})\cos\varphi + (\boldsymbol{n}\times\boldsymbol{\sigma})\sin\varphi\,.$$

End Solution

(d) Consider the special case $\boldsymbol{n} = \boldsymbol{e}_z$ and discuss infinitesimal rotations.

Solution

In this case,

$$U \to 1 + i\sigma_z\delta\varphi/2 = \begin{pmatrix} 1 + i\delta\varphi/2 & 0 \\ 0 & 1 - i\delta\varphi/2 \end{pmatrix}\,,$$

and

$$\boldsymbol{\sigma}' = U\boldsymbol{\sigma}U^\dagger \to \boldsymbol{\sigma} - \boldsymbol{e}_x\sigma_y\delta\varphi + \boldsymbol{e}_y\sigma_x\delta\varphi\,.$$

The rotation about the z-axis carries a little bit of the x-component of the Pauli operator into the y-direction and a little bit of the y-component into the negative x-direction, just as what happens with an ordinary vector under rotation.

End Solution

(e) For a spinor

$$\chi = \begin{pmatrix} \alpha_+ \\ \alpha_- \end{pmatrix}\,,$$

calculate the transformed spinor χ' for $\boldsymbol{n} = \boldsymbol{e}_z$.

Solution

$$\begin{aligned}\chi' = U\chi &= (\cos\varphi/2 + i\sigma_z\sin\varphi/2)\begin{pmatrix} \alpha_+ \\ \alpha_- \end{pmatrix} \\ &= \left(\cos\varphi/2\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + i\sin\varphi/2\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\right)\begin{pmatrix} \alpha_+ \\ \alpha_- \end{pmatrix} \\ &= \begin{pmatrix} \alpha_+\cos\varphi/2 + i\alpha_+\sin\varphi/2 \\ \alpha_-\cos\varphi/2 - i\alpha_-\sin\varphi/2 \end{pmatrix} = \begin{pmatrix} \alpha_+ e^{+i\varphi/2} \\ \alpha_- e^{-i\varphi/2} \end{pmatrix}\,.\end{aligned}$$

End Solution

(f) What happens to the spinor when $\varphi = 2\pi$?

Solution

The spinor changes sign! A rotation of 4π is required to bring the spinor back to its original state! This is a weird property of fermions. However, expectation values involve $|\chi|^2$ so the sign cancels out!

End Solution

2. Spin precession.

(a) The spinors

$$|\uparrow\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \qquad |\downarrow\rangle = \begin{pmatrix} 0 \\ 1 \end{pmatrix} ,$$

are the eigenfunctions of σ_z. They are an orthonormal, complete basis in which σ_z is diagonal. In this same basis, what are the eigenfunctions (spinors) of σ_x and σ_y?

Solution

The eigenvalues are ± 1, so the expression for the eigenspinors of σ_x is

$$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} a \\ b \end{pmatrix} = \pm \begin{pmatrix} a \\ b \end{pmatrix} ,$$

which gives $b = \pm a$. In order that the spinors be normalized, we take $|a| = |b| = 1/\sqrt{2}$. Then the eigenspinors of σ_x are

$$|\uparrow_x\rangle = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 1 \end{pmatrix} \qquad |\downarrow_x\rangle = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ -1 \end{pmatrix} .$$

For σ_y,

$$\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \begin{pmatrix} a \\ b \end{pmatrix} = \pm \begin{pmatrix} a \\ b \end{pmatrix} ,$$

which gives $b = \pm ia$. Again, we must take $|a| = |b| = 1/\sqrt{2}$ and the eigenspinors of σ_y are

$$|\uparrow_y\rangle = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ i \end{pmatrix} \qquad |\downarrow_y\rangle = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ -i \end{pmatrix} .$$

End Solution

Now we consider a spin 1/2 particle in a uniform magnetic field. For definiteness, we suppose it's an electron, so the Hamiltonian for the spin degree of freedom is

$$H = \frac{ge}{2mc} \boldsymbol{S} \cdot \boldsymbol{B} \approx \frac{e}{mc} \boldsymbol{S} \cdot \boldsymbol{B} .$$

(b) The state of the system is represented by the spinor $\chi = \begin{pmatrix} \chi_+ \\ \chi_- \end{pmatrix}$. What is the equation of motion for χ? After writing down the equation for the general case, specialize to the case $\boldsymbol{B} = B\boldsymbol{e}_z$. Your results should depend on the frequency $\omega = eB/mc$ which will turn out to be the precession frequency.

Solution

It's just the Schroedinger equation (also known as the Pauli equation in this case).

$$i\hbar\frac{\partial}{\partial t}\begin{pmatrix} \chi_+ \\ \chi_- \end{pmatrix} = \frac{e}{mc}\boldsymbol{S}\cdot\boldsymbol{B}\begin{pmatrix} \chi_+ \\ \chi_- \end{pmatrix} = \mu_B\boldsymbol{\sigma}\cdot\boldsymbol{B}\begin{pmatrix} \chi_+ \\ \chi_- \end{pmatrix} = \mu_B\begin{pmatrix} B_z & B_x - iB_y \\ B_x + iB_y & -B_z \end{pmatrix}\begin{pmatrix} \chi_+ \\ \chi_- \end{pmatrix} .$$

When $\boldsymbol{B} = B\boldsymbol{e}_z$, the equation of motion becomes

$$i\frac{\partial}{\partial t}\begin{pmatrix} \chi_+ \\ \chi_- \end{pmatrix} = \frac{\mu_B B}{\hbar}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\begin{pmatrix} \chi_+ \\ \chi_- \end{pmatrix} = \frac{\omega}{2}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\begin{pmatrix} \chi_+ \\ \chi_- \end{pmatrix} ,$$

where $\omega = 2\mu_B B/\hbar = eB/(mc)$ is the frequency (which will turn out to the precession frequency).

End Solution

(c) In the case with $\boldsymbol{B} = B\boldsymbol{e}_z$, and in the Heisenberg representation, what is the time dependence of the spin operator, $\boldsymbol{S}(t)$?

Solution

In the Heisenberg representation,

$$\frac{dS_z}{dt} = \frac{i}{\hbar}[H, S_z] = 0 ,$$

so $S_z(t) = S_z(0)$. For the other two components, it's more convenient to work with $S_x \pm iS_y$.

$$\begin{aligned}
\frac{d}{dt}(S_x \pm iS_y) &= \frac{i}{\hbar}[H, S_x \pm iS_y] \\
&= \frac{ieB}{mc\hbar}[S_z, S_x \pm iS_y] \\
&= \frac{ieB}{mc\hbar}i\hbar(S_y \mp iS_x) \\
&= \pm\frac{ieB}{mc}(S_x \pm iS_y) .
\end{aligned}$$

This means

$$S_x(t) \pm iS_y(t) = e^{\pm i\omega t}(S_x(0) \pm iS_y(0)) ,$$

or

$$\begin{aligned}
S_x(t) &= \cos\omega t\, S_x(0) - \sin\omega t\, S_y(0) \\
S_y(t) &= \sin\omega t\, S_x(0) + \cos\omega t\, S_y(0) .
\end{aligned}$$

End Solution

(d) In the case with $\boldsymbol{B} = B\boldsymbol{e}_z$, and in the Schroedinger representation, what is the time dependence of the spinor, $\chi(t)$?

Solution

$$\chi(t) = e^{-iHt/\hbar}\chi(0)\,.$$

The argument of the exponential is

$$\frac{-iHt}{\hbar} = \frac{-ieBt}{mc\hbar}S_z = \frac{-i\omega t}{\hbar}S_z = i\varphi\boldsymbol{n}\cdot\boldsymbol{S}/\hbar\,,$$

provided we identify $\varphi = -\omega t$ and $\boldsymbol{n} = \boldsymbol{e}_z$. This form is exactly the form for the unitary operator discussed in problem 1, so the time dependence of χ is

$$\begin{aligned}\chi(t) = \begin{pmatrix}\chi_+(t)\\ \chi_-(t)\end{pmatrix} &= \left(\cos(-\omega t/2)\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix} + i\sin(-\omega t/2)\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}\right)\begin{pmatrix}\chi_+(0)\\ \chi_-(0)\end{pmatrix}\\ &= \begin{pmatrix}e^{-i\omega t/2} & 0\\ 0 & e^{+i\omega t/2}\end{pmatrix}\begin{pmatrix}\chi_+(0)\\ \chi_-(0)\end{pmatrix}\\ &= \begin{pmatrix}e^{-i\omega t/2}\chi_+(0)\\ e^{+i\omega t/2}\chi_-(0)\end{pmatrix}\,.\end{aligned}$$

This makes sense. The up and down states are the stationary states of the Hamiltonian with energy $\hbar\omega/2$ for the up state. The spin in the up state is aligned with $\boldsymbol{B}$, but the magnetic moment is anti-aligned, so the energy is higher than in the down state which has energy $-\hbar\omega/2$.

End Solution

(e) At $t = 0$, the spin is aligned along the x axis. What is the probability of getting $\hbar/2$ in a measurement of S_z at time t.

Solution

The initial state must be (to within a phase!)

$$\chi(0) = |\uparrow_x\rangle = \frac{1}{\sqrt{2}}\begin{pmatrix}1\\ 1\end{pmatrix}\,.$$

From part (e), we find the state at time t and project that onto $\langle\uparrow_z|$ to find the amplitude for going to spin up along z, and then square it to get the probability.

$$P(\uparrow_z) = \left|\begin{pmatrix}1 & 0\end{pmatrix}\begin{pmatrix}e^{-i\omega t/2}/\sqrt{2}\\ e^{+i\omega t/2}\sqrt{2}\end{pmatrix}\right|^2 = \frac{1}{2}\,,$$

which shouldn't be a surprise. If the spin is pointed along the x axis, it's equally likely to be up or down if a measurement is made along the z axis. When the spin is in a magnetic field in the z-direction it precesses about the z-axis which does not change its z-component (or the probability distribution of its z-component).

End Solution

(f) At $t = 0$, the spin is aligned along the x axis. What is the probability of getting $\hbar/2$ in a measurement of S_x at time t.

Solution

We solve this part the same way, except we project onto $\langle \uparrow_x |$.

$$P(\uparrow_x) = \left| \begin{pmatrix} 1/\sqrt{2} & 1/\sqrt{2} \end{pmatrix} \begin{pmatrix} e^{-i\omega t/2}/\sqrt{2} \\ e^{+i\omega t/2}\sqrt{2} \end{pmatrix} \right|^2 = \cos^2 \omega t/2 = (1 + \cos(\omega t))/2 \,,$$

so the spin is, in fact, precessing with a frequency ω!

The solutions to parts (e) and (f) have been based on the Schroedinger representation. In the Heisenberg representation, we could ask about the expectation value of $S_x(t)$. Recall that the state does not change. We use $S_x(t)$ from part (c).

$$\langle S_x(t) \rangle = \begin{pmatrix} 1/\sqrt{2} & 1/\sqrt{2} \end{pmatrix} \begin{pmatrix} 0 & \hbar e^{i\omega t}/2 \\ \hbar e^{-i\omega t}/2 & 0 \end{pmatrix} \begin{pmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \end{pmatrix} = \frac{\hbar}{2} \cos \omega t \,,$$

which is consistent with the idea that the spin is precessing around the z-axis, which means its projection on the x-axis oscillates between $\pm\hbar/2$.

End Solution

3. Magnetic Resonance. This problem continues from where problem 2 ended. We have an electron in a uniform magnetic field along the z-axis. In addition, we have a small, time variable field along the x-axis: $B_x = B_p \cos \omega t$, where ω is determined by the field in the z-direction, $\omega = eB/mc$. We suppose that at $t = 0$, the electron has spin down along the z-axis. The experimental picture is that the magnetic moment of the electron is aligned with the field (so the spin is anti-aligned) and the "probe" field is turned on at $t = 0$. What happens? In particular what is $\langle S_z(t) \rangle$? Use the interaction representation to solve this problem.

Hint: The probe field is oscillating along the x-axis. However you can think of it as the sum of two fields, each rotating in the xy-plane with angular velocity ω. One rotates in the positive direction and one rotates in the negative direction. When you transform the Hamiltonian due to the probe field to the interaction representation, one of these will wind up stationary, and the other will wind up rotating at 2ω. Make an argument for why the 2ω term can be ignored and then proceed with the stationary term.

Solution

The additional term in the Hamiltonian is

$$V(t) = \frac{eB_p}{mc} S_x \cos\omega t = 2\omega_p S_x \cos\omega t \,.$$

In the interaction picture, the operators evolve just as they did in the Heisenberg picture, so we can use problem 2, part (c) for the $\boldsymbol{S}(t)$. In particular $S_z(t) = S_z(0)$, so that was easy!

Transforming the extra term in the Hamiltonian requires a bit more work. First, as mentioned in the hint, we can write the extra magnetic field as the sum of $\boldsymbol{B}_+ + \boldsymbol{B}_-$ with

$$\boldsymbol{B}_\pm = \frac{B_p}{2} (\cos\omega t \, \boldsymbol{e}_x \pm \sin\omega t \, \boldsymbol{e}_y) \,,$$

Then the extra term is $V_+ + V_-$ with

$$V_\pm = \frac{eB_p}{2mc} (S_x \cos\omega t \pm S_y \sin\omega t) \,.$$

Now we must transform this extra term to the Heisenberg representation based on the Hamiltonian without V. Again, we use problem 2(c) and replace $S_x(t)$ by its expression in terms of $S_x(0)$ and $S_y(0)$ and similarly for $S_y(t)$.

$$V_{I\pm}(t) = \omega_p \left(\cos^2\omega t S_x(0) - \cos\omega t \sin\omega t S_y(0) \pm \sin^2\omega t S_x(0) \pm \cos\omega t \sin\omega t S_y(0) \right) \,.$$

So,

$$V_{I+}(t) = \omega_p S_x(0) \,,$$

$$V_{I-}(t) = \omega_p \left(\cos 2\omega t S_x(0) - \sin 2\omega t S_y(0) \right) \,.$$

The integral of the Hamiltonian with time gives the phase of the wave function. V_{I-} is oscillating (about 0) twice as fast as anything else, so it essentially never has time to build up an appreciable phase. Thus we ignore it and continue with $V_{I+}(t)$ which is just a constant.

The state evolves as states in the Schroedinger representation, but with the interaction Hamiltonian rather than the full Hamiltonian.

$$i\hbar \frac{\partial}{\partial t} \chi = V_{I+}(t) \chi$$

$$\chi(t) = e^{-iV_{I+}(t)/\hbar} \chi(0) = e^{-i\omega_p \boldsymbol{e}_x \cdot \boldsymbol{S}(0) t/\hbar} \chi(0) \,.$$

Again, we make use of problem 1 to write this as

$$\chi(t) = \left(\cos(-\omega_p t/2) + i\sigma_x(0) \sin(-\omega_p t/2) \right) \chi(0) \,.$$

The state at $t = 0$ is spin down along the z-axis, so

$$\begin{pmatrix} \chi_+ \\ \chi_- \end{pmatrix} = \begin{pmatrix} \cos\omega_p t/2 & -i\sin\omega_p t/2 \\ -i\sin\omega_p t/2 & \cos\omega_p t/2 \end{pmatrix} \begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} -i\sin\omega_p t/2 \\ \cos\omega_p t/2 \end{pmatrix} .$$

So we have found how the state evolves due to the probe field and we know how S_z evolves due to the main field, and we can calculate $\langle S_z \rangle$

$$\langle S_z(t)\rangle = (\, +i\sin\omega_p t/2 \quad \cos\omega_p t/2 \,) \begin{pmatrix} \hbar/2 & 0 \\ 0 & -\hbar/2 \end{pmatrix} \begin{pmatrix} -i\sin\omega_p t/2 \\ \cos\omega_p t/2 \end{pmatrix} = -\frac{\hbar}{2}\cos\omega_p t\,.$$

After all that, we have a very simple answer! The spin flips up and down with frequency ω_p. Physically, the Heisenberg representation, in this case, corresponds to a frame rotating with the precessing spin. Since we made the probe field oscillate with the same frequency as the rotation, it split into two pieces, one that's stationary in the rotating frame and one that's rotating like mad and has no net effect. Since the field is stationary in the rotating frame, it causes the spin to precess about this field in the rotating frame and it oscillates between aligned and anti-aligned. You have just discovered the principle of magnetic resonance imaging (which generally uses nuclear spins, but the idea is the same). A sample is placed in a strong field, this causes the spins to precess. Because the environment of the spins depends not only on the applied field, but the field due to neighbors, the actual precession frequency will vary with chemical environment. The probe field is scanned in frequency (slowly compared to ω and ω_p) and there will be an absorption and emission of energy by the sample as the resonant frequency is passed. The strength of the signal depends on the density of aligned spins. A lot can be learned about the sample by studying how the spins precess and flip!

——— End Solution ———

4. Combining angular momentum. Two electron spins, $\boldsymbol{S}_1$ and $\boldsymbol{S}_2$ can be summed to produce a total angular momentum $\boldsymbol{J} = \boldsymbol{S}_1 + \boldsymbol{S}_2$. The states $|j\,m\,s_1\,s_2\rangle$ can be expanded in the states $|s_1\,m_1\rangle\,|s_2\,m_2\rangle$. Determine the expansion.

——— Solution ———

Addition of two spin 1/2 angular momenta produces a spin 1 system with $j = 1$ and $m = -1,\ 0,\ +1$, (the triplet) and a spin system with $j = 0$ and $m = 0$. So,

$$|1, +1, 1/2, 1/2\rangle = |1/2, 1/2\rangle\,|1/2, 1/2\rangle\ .$$

Recall that

$$L_\pm\,|l\,m\rangle = \hbar\sqrt{l(l+1) - m(m\pm1)}\,|l, m\pm1\rangle\ .$$

We operate with $J_- = S_{1-} + S_{2-}$ to get

$$\begin{aligned} \sqrt{1(1+1) - 1(1-1)}\,|1, 0, 1/2, 1/2\rangle &= \\ &\sqrt{(1/2)(1/2+1) - (1/2)(1/2-1)}\,|1/2, -1/2\rangle\,|1/2, 1/2\rangle \\ &+\sqrt{(1/2)(1/2+1) - (1/2)(1/2-1)}\,|1/2, 1/2\rangle\,|1/2, -1/2\rangle\ , \end{aligned}$$

or

$$|1,0,1/2,1/2\rangle = \frac{1}{\sqrt{2}}\left(|1/2,-1/2\rangle\,|1/2,1/2\rangle + |1/2,1/2\rangle\,|1/2,-1/2\rangle\right)\ .$$

If we operate with J_- again, we find

$$|1,-1,1/2,1/2\rangle = |1/2,-1/2\rangle\,|1/2,-1/2\rangle\ .$$

Finally,

$$|0,0,1/2,1/2\rangle = \frac{1}{\sqrt{2}}\left(|1/2,-1/2\rangle\,|1/2,1/2\rangle - |1/2,1/2\rangle\,|1/2,-1/2\rangle\right)\ .$$

We find this by noting that it must be orthogonal to $|1,0,1/2,1/2\rangle$ and it must contain the same m_1 and m_2 states that add up to 0.

Comment: we will learn later that the wave functions for Fermions must be anti-symmetric (that is, change sign), when any two Fermions are exchanged. The singlet state above is anti-symmetric, so this must be combined with a symmetric state for the spatial wave function to have a suitable overall state. Similarly, the triplet state must be combined with an anti-symmetric spatial wave function.

——————— End Solution ———————