1. Angular momentum uncertainty relations. A system is in the lm eigenstate of L^2, L_z.

(a) Show that the expectation values of $L_\pm = L_x \pm iL_y$, L_x, and L_y all vanish.

Solution

$\langle \psi_{lm} \mid L_\pm \mid \psi_{lm} \rangle = \hbar\sqrt{l(l+1) - m(m \pm 1)}\, \langle \psi_{lm} \mid \psi_{l,m\pm 1} \rangle = 0$, since ψ_{lm} $\psi_{l,m\pm 1}$ are orthogonal. Note in the special cases $m = \pm l$ for $L_\pm$, the square root kills the expectation value! $L_x = (L_+ + L_-)/2$ and $L_y = -i(L_+ - L_-)/2$, so these expectations must be zero as well.

End Solution

(b) Determine $\Delta L_i = \sqrt{\langle L_i^2 \rangle - \langle L_i \rangle^2}$. Verify the generalized uncertainty relation holds for all pairs of angular momentum components. Comment on $\Delta L_x \Delta L_y$ for the cases $m = 0$ and $m = l$.

Solution

$\langle L_z \rangle = m\hbar$, $\left\langle L_z^2 \right\rangle = m^2\hbar^2$, $\Delta L_z = 0$. The state ψ_{lm} is symmetric in x and y, so $\left\langle L_x^2 \right\rangle = \left\langle L_y^2 \right\rangle = \left(\left\langle L^2 \right\rangle - \left\langle L_z^2 \right\rangle\right)/2 = \hbar^2(l(l+1) - m^2)/2$. $\Delta L_x = \Delta L_y = \hbar\sqrt{(l(l+1) - m^2)/2}$.

$$\Delta L_x \Delta L_z = 0 \geq |\, \langle [L_x, L_z] \rangle \,|/2 = |\, \langle -i\hbar L_y \rangle \,| = 0\,.$$

This is not very exciting! Similarly, $\Delta L_y \Delta L_z = 0$, also not exciting. However,

$$\Delta L_x \Delta L_y = \hbar^2(l(l+1) - m^2)/2 \geq |\, \langle [L_x, L_y] \rangle \,|/2 = |\, \langle i\hbar L_z \rangle \,| = \hbar^2 m/2\,.$$

In the case $m = 0$, there is no angular momentum about the z-axis. The particle is found as close to the z-axis as possible. In this case the minimum uncertainty product for the x and y components is 0 and this is well satisfied since all the angular momentum is in the x and y components (but with vanishing expectation value). The only case where the equality sign would be used in the uncertainty relation is the case $l = 0$ in which case all components and their squares are zero.

The case $m = l$ corresponds to the maximum angular momentum component along the z-axis. One might visualize the particle in the xy-plane rotating about the z-axis. Of course, it can't be exactly in the xy-plane and its out of plane motion produces some components of L_x and L_y which average to 0, but have some spread around the average. The uncertainty relation becomes

$$\Delta L_x \Delta L_y = \hbar^2(l(l+1) - l^2)/2 = \hbar^2 l/2 = |\, \langle [L_x, L_y] \rangle \,|/2 = |\, \langle i\hbar L_z \rangle \,| = \hbar^2 l/2\,.$$

In this case, the angular momentum components in the x and y directions have the minimum possible uncertainty product.

End Solution

2. Fun with angular momentum commutators.

(a) Suppose the vector operators $\boldsymbol{A}$ and $\boldsymbol{B}$ commute with each other and $\boldsymbol{L}$. Show that

$$[\boldsymbol{A}\cdot\boldsymbol{L}, \boldsymbol{B}\cdot\boldsymbol{L}] = i\hbar(\boldsymbol{A}\times\boldsymbol{B})\cdot\boldsymbol{L}\,.$$

Solution

We use the summation convention and the completely anti-symmetric tensor and just grind it out!

$$\begin{aligned}
[\boldsymbol{A}\cdot\boldsymbol{L}, \boldsymbol{B}\cdot\boldsymbol{L}] &= A_i L_i B_j L_j - B_j L_j A_i L_i \\
&= A_i B_j (L_i L_j - L_j L_i) \\
&= A_i B_j [L_i, L_j] \\
&= A_i B_j (i\hbar\epsilon_{ijk} L_k) \\
&= -A_i B_j (i\hbar\epsilon_{ikj} L_k) \\
&= +A_i B_j (i\hbar\epsilon_{kij} L_k) \\
&= i\hbar(\boldsymbol{A}\times\boldsymbol{B})_k L_k \\
&= i\hbar(\boldsymbol{A}\times\boldsymbol{B})\cdot\boldsymbol{L}\,.
\end{aligned}$$

End Solution

(b) Suppose $\boldsymbol{V}$ is a vector operator which might be a function of $\boldsymbol{x}$ and $\boldsymbol{p}$, so it doesn't necessarily commute with $\boldsymbol{L}$. Show that

$$[\boldsymbol{L}^2, \boldsymbol{V}] = 2i\hbar(\boldsymbol{V}\times\boldsymbol{L} - i\hbar\boldsymbol{V})\,.$$

You might need the relation derived in lecture for a vector operator, $[L_i, V_j] = i\hbar\epsilon_{ijk}V_k$.

Solution

Again, grinding it out works:

$$\begin{aligned}
\left([\boldsymbol{L}^2, \boldsymbol{V}]\right)_i &= L_j L_j V_i - V_i L_j L_j \\
&= L_j V_i L_j + L_j [L_j, V_i] - L_j V_i L_j + [L_j, V_i] L_j \\
&= i\hbar\left(L_j \epsilon_{jik} V_k + \epsilon_{jik} V_k L_j\right) \\
&= i\hbar\left(\epsilon_{jik} L_j V_k + \epsilon_{jik} V_k L_j\right) \\
&= i\hbar\left(\epsilon_{jik} V_k L_j + \epsilon_{jik}[L_j, V_k] + \epsilon_{jik} V_k L_j\right) \\
&= i\hbar\left(2\epsilon_{jik} V_k L_j + i\hbar\epsilon_{jik}\epsilon_{jkl} V_l\right) \\
&= i\hbar\left(-2\epsilon_{ijk} V_k L_j - i\hbar\epsilon_{ijk}\epsilon_{jkl} V_l\right) \\
&= i\hbar\left(+2\epsilon_{ikj} V_k L_j + i\hbar\epsilon_{ijk}\epsilon_{jlk} V_l\right) \\
&= i\hbar\left(+2\epsilon_{ikj} V_k L_j - i\hbar\epsilon_{ijk}\epsilon_{ljk} V_l\right) \\
&= i\hbar\left(+2\epsilon_{ikj} V_k L_j - 2i\hbar\delta_{il} V_l\right) \\
&= i\hbar\left(+2\epsilon_{ikj} V_k L_j - 2i\hbar V_i\right) \\
&= 2i\hbar\left(\boldsymbol{V}\times\boldsymbol{L} - i\hbar\boldsymbol{V}\right)_i\,.
\end{aligned}$$

Note that we used the fact that $\epsilon_{ijk}\epsilon_{ljk} = 2\delta_{il}$. This is easy to see. Pick an i, say 1. Then the only non-zero values of ϵ_{ijk} are those with $j, k = 2, 3$ or $j, k = 3, 2$. In the second tensor, the only non-zero values will occur for $l = 1$, the sign will be the same as the first, and there are two contributions.

End Solution

3. Classically, a particle moving in a spherically symmetric potential has the Hamiltonian

$$H = \frac{p_r^2}{2m} + \frac{L^2}{2mr^2} + V(r)\,,$$

where $p_r = \boldsymbol{r}\cdot\boldsymbol{p}/r$. For quantum mechanics, we must define

$$p_r = \frac{1}{2}\left(\frac{1}{r}(\boldsymbol{r}\cdot\boldsymbol{p}) + (\boldsymbol{p}\cdot\boldsymbol{r})\frac{1}{r}\right)\,, \tag{1}$$

with the Hermitian operator

$$p_r = \frac{\hbar}{i}\left(\frac{\partial}{\partial r} + \frac{1}{r}\right)\,. \tag{2}$$

Show the operator defined in equation (1) is the same as that in equation (2). Show that p_r in equation (2) is Hermitian (consider $\psi(r,\theta,\phi)$ and $\varphi(r,\theta,\phi)$) and that when used in the Hamilton, p_r of equation (2) gives the correct Schroedinger equation. Also show that the operator $(\hbar/i)(\partial/\partial r)$ is not Hermitian!

Solution

$\boldsymbol{r} = r\boldsymbol{e}_r$. The momentum operator in spherical coordinates is

$$\frac{\hbar}{i}\nabla = \frac{\hbar}{i}\left(\boldsymbol{e}_r\frac{\partial}{\partial r} + \boldsymbol{e}_\theta\frac{1}{r}\frac{\partial}{\partial\theta} + \boldsymbol{e}_\phi\frac{1}{r\sin\theta}\frac{\partial}{\partial\phi}\right)\,.$$

The first term in equation (1) becomes $(\hbar/2i)(\partial/\partial r)$. The second term is a bit more work. It's most easily evaluated in a mix of Cartesian and spherical coordinates. Note that the momentum operator in the second term operates on $\boldsymbol{r}$, $1/r$, and whatever might be to the right. When it operates on whatever is to the right, we get a term that's the same as the first term. So let's just evaluate the second term when the $\boldsymbol{p}$ operates on $\boldsymbol{r}$ and $1/r$.

$$\begin{aligned}
\frac{1}{2}\boldsymbol{p}\cdot\left(\frac{\boldsymbol{r}}{r}\right) &= \frac{\hbar}{2i}\left(\frac{\partial}{\partial x}\frac{x}{r} + \frac{\partial}{\partial y}\frac{y}{r} + \frac{\partial}{\partial x}\frac{z}{r}\right)\\
&= \frac{\hbar}{2i}\left(\frac{1}{r} - \frac{x^2}{r^3} + \frac{1}{r} - \frac{y^2}{r^3} + \frac{1}{r} - \frac{z^2}{r^3}\right)\\
&= \frac{\hbar}{2i}\left(\frac{3}{r} - \frac{x^2+y^2+z^2}{r^3}\right)\\
&= \frac{\hbar}{i}\frac{1}{r}\,,
\end{aligned}$$

and putting this together with the two other pieces, we see that indeed, the operator in equation (1) is the same as that in equation (2).

The adjoint of p_r is

$$p_r^\dagger = \frac{\hbar}{-i}\left(\frac{\partial}{\partial r} + \frac{1}{r}\right) .$$

We consider

$$\langle p_r\varphi \mid \psi\rangle = \int_0^\infty r^2\,dr \int_0^\pi \sin\theta\,d\theta \int_0^{2\pi} d\phi\,\left(-\frac{\hbar}{i}\left(\frac{\partial}{\partial r} + \frac{1}{r}\right)\varphi^*(r,\theta,\phi)\right)\psi(r,\theta,\phi)\,.$$

The angular integrations can be done independently of the radial integration, so we just assume they have been done (or will be done) and drop the angular variables from the discussion. We'll use an integration by parts to move the derivative from the φ to ψ. The derivative also acts on r^2.

$$\begin{aligned}
\langle p_r\varphi \mid \psi\rangle &= \int_0^\infty \left(-\frac{\hbar}{i}\left(\frac{\partial}{\partial r} + \frac{1}{r}\right)\varphi^*(r)\right)\psi(r)\,r^2\,dr \\
&= -\frac{\hbar}{i}\left(\int_0^\infty \frac{\partial\varphi^*}{\partial r}\psi\,r^2\,dr + \int_0^\infty \frac{1}{r}\varphi^*\psi\,r^2\,dr\right) \\
&= -\frac{\hbar}{i}\left(\varphi*\psi\,r^2\Big|_0^\infty - \int_0^\infty \varphi^*\frac{\partial\psi}{\partial r}\,r^2\,dr - \int_0^\infty \varphi^*\psi\,(2r)\,dr + \int_0^\infty \frac{1}{r}\varphi^*\psi\,r^2\,dr\right) \\
&= \frac{\hbar}{i}\left(0 + \int_0^\infty \varphi^*\frac{\partial\psi}{\partial r}\,r^2\,dr + \int_0^\infty \varphi^*\,\frac{2}{r}\,\psi\,r^2\,dr - \int_0^\infty \varphi^*\frac{1}{r}\psi\,r^2\,dr\right) \\
&= \int_0^\infty \varphi^*\,\left(\frac{\hbar}{i}\left(\frac{\partial}{\partial r} + \frac{1}{r}\right)\psi\right)\,r^2\,dr \\
&= \langle \varphi \mid p_r\psi\rangle\ ,
\end{aligned}$$

and p_r is Hermitian. Note that the evaluation of the limits gives 0 since the integrand must converge at both ends!

If we do the same calculation with $p_{r\,\mathrm{bogus}} = (\hbar/i)(\partial/\partial r)$ we find

$$\langle p_{r\,\mathrm{bogus}}\varphi \mid \psi\rangle - \langle \varphi \mid p_{r\,\mathrm{bogus}}\psi\rangle = \frac{\hbar}{i}\left\langle \varphi \,\middle|\, \frac{2}{r}\,\psi\right\rangle\ ,$$

which is not zero in general, so $p_{r\,\mathrm{bogus}}$ is not Hermitian.

Finally, the Laplacian, in spherical coordinates, is

$$\nabla^2 = \frac{1}{r^2}\frac{\partial}{\partial r}\,r^2\,\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\,\sin\theta\,\frac{\partial}{\partial\theta} + \frac{1}{r^2\sin^2\theta}\frac{\partial^2}{\partial\phi^2}\,.$$

If we multiply this by $-\hbar^2$, we get p^2 for the Schroedinger equation expressed in spherical coordinates. We've already seen in lecture that the angular part is L^2/r^2. So it remains to show that the first term times $-\hbar^2$ is the same as p_r^2. The first term is

$$-\hbar^2 \frac{1}{r^2}\frac{\partial}{\partial r} r^2 \frac{\partial}{\partial r} = -\hbar^2 \left(\frac{\partial^2}{\partial r^2} + \frac{2}{r}\frac{\partial}{\partial r} \right) .$$

Squaring p_r,

$$\begin{aligned} p_r^2 &= -\hbar^2 \left(\frac{\partial}{\partial r} + \frac{1}{r} \right) \left(\frac{\partial}{\partial r} + \frac{1}{r} \right) \\ &= -\hbar^2 \left(\frac{\partial^2}{\partial r^2} - \frac{1}{r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{1}{r^2} \right) \\ &= -\hbar^2 \left(\frac{\partial^2}{\partial r^2} + \frac{2}{r}\frac{\partial}{\partial r} \right) . \end{aligned}$$

End Solution

4. What if we like x instead of z?

(a) Find the eigenfunction, ψ, of L^2 and L_x with eigenvalues $2\hbar^2$ and $\hbar$, respectively.

Solution

Since the eigenvalue of L^2 is $2\hbar^2$, the eigenfunction has $l = 1$. The eigenfunctions of L_z with $l = 1$ are

$$\begin{aligned} Y_{1,+1} &= -\sqrt{\frac{3}{8\pi}} \sin\theta \, e^{+i\phi} &&= -\sqrt{\frac{3}{8\pi}} \frac{1}{r}(x + iy) \\ Y_{1,0} &= +\sqrt{\frac{3}{4\pi}} \cos\theta &&= +\sqrt{\frac{3}{4\pi}} \frac{1}{r} z \\ Y_{1,-1} &= +\sqrt{\frac{3}{8\pi}} \sin\theta \, e^{-i\phi} &&= +\sqrt{\frac{3}{8\pi}} \frac{1}{r}(x - iy) . \end{aligned}$$

If we rotate the coordinate system about the y-axis so the new x-axis is along the old z-axis then there will be one unit of angular momentum along the new x-axis. This means replacing z by x, y by y, and x by $-z$. With these replacements, the eigenfunction $Y_{1,+1}$ will have one unit of angular momentum along the x-axis (and of course, it will still be an eigenfunction of L^2 with eigenvalue $2\hbar^2$. The desired eigenfunction is then

$$\psi = -\sqrt{\frac{3}{8\pi}} \frac{1}{r}(-z + iy) = -\sqrt{\frac{3}{8\pi}} (-\cos\theta + i \sin\theta \sin\phi) .$$

We can check this by operating with L_x in polar coordinates.

$$\begin{aligned}
L_x\psi &= \frac{\hbar}{i}\left(-\sin\phi\frac{\partial}{\partial\theta} - \cos\phi\cot\theta\frac{\partial}{\partial\phi}\right)\left(-\sqrt{\frac{3}{8\pi}}\,(-\cos\theta + i\sin\theta\sin\phi)\right)\\
&= -\frac{\hbar}{i}\sqrt{\frac{3}{8\pi}}\,\left(-\sin\phi\sin\theta - i\sin^2\phi\cos\theta - i\cos\theta\cos^2\phi\right)\\
&= \hbar\left(-\sqrt{\frac{3}{8\pi}}\,(-\cos\theta + i\sin\theta\sin\phi)\right)\\
&= \hbar\psi\,.
\end{aligned}$$

End Solution

(b) Express the ψ just found as a linear combination of eigenfunctions of L^2 and L_z.

Solution

Of course, if we want to write

$$\psi(\theta,\phi) = \sum_{l=0}^{\infty}\sum_{m=-l}^{+l} c_{lm}Y_{lm}(\theta,\phi)\,,$$

we find the c_{lm} by projecting ψ onto the Y_{lm},

$$c_{lm} = \int_0^{\pi}\sin\theta\,d\theta\int_0^{2\pi}d\phi\,Y^*_{lm}(\theta,\phi)\,\psi(\theta,\phi)\,,$$

but in this case, it's easier just to look at ψ and the $Y_{1,+1}$, $Y_{1,0}$, and $Y_{1,-1}$ and decide what we need:

$$\psi = \frac{1}{2}Y_{1,+1} + \frac{1}{\sqrt{2}}Y_{1,0} + \frac{1}{2}Y_{1,-1}\,.$$

The sum of the squares of the coefficients is 1, as it should be.

End Solution

5. Algebra all the way. We used algebraic techniques in lecture to deduce that L^2 and L_z could be simultaneously diagonalized (that is, eigenfunctions could be eigenfunctions of L^2 and L_z at the same time) and that the eigenvalues are $l(l+1)\hbar^2$ and $m\hbar$, respectively, with l and m integers or half integers and with $l \geq 0$ and $m = -l, -l+1, \ldots l-1, l$. We then abandoned the algebraic technique and solved a differential equation to find the orbital angular momentum eigenfunctions. Here we will outline the use of algebraic techniques to deduce the orbital angular momentum eigenfunctions. We start by introducing $x_\pm = x \pm iy$.

(a) Show that the following commutation relations hold (you may use relations already derived in lecture):

$$
\begin{aligned}
[L_z, x_\pm] &= \pm\hbar x_\pm \\
[L_\pm, x_\pm] &= 0 \\
[L_\pm, x_\mp] &= \pm 2\hbar z \\
[L^2, x_+] &= 2\hbar x_+ L_z + 2\hbar^2 x_+ - 2\hbar z L_+ \,.
\end{aligned}
$$

Solution

$[L_z, x] = i\hbar y$ and $[L_z, iy] = (-i)(i)\hbar x = \hbar x$, so $[L_z, x \pm iy] = \hbar(\pm x + iy) = \pm\hbar x_\pm$.

$[L_\pm, x_\pm] = [L_x \pm iL_y, x \pm iy] = \pm i[L_x, y] \pm i[L_y, x] = -(\pm\hbar z) + (\pm\hbar z) = 0.$

$[L_\pm, x_\mp] = [L_x \pm iL_y, x \mp iy] = \mp i[L_x, y] \pm i[L_y, x] = +(\pm\hbar z) + (\pm\hbar z) = \pm 2\hbar z.$

For the final commutator, we use the following,

$$[AB, C] = ABC - BAC = ACB + A[B, C] - [C, A]B - ACB = A[B, C] + [A, C]B \,.$$

Also, from lecture

$$L^2 = L_- L_+ + \hbar L_z + L_z^2 \,.$$

So

$$[L_- L_+, x_+] = L_-[L_+, x_+] + [L_-, x_+]L_+ = 0 - 2\hbar z L_+ \,,$$

$$\hbar[L_z, x_+] = \hbar^2 x_+ \,,$$

$$[L_z^2, x_+] = L_z[L_z, x_+] + [L_z, x_+]L_z = \hbar L_z x_+ + \hbar x_+ L_z = \hbar x_+ L_z + \hbar^2 x_+ + \hbar x_+ L_z \,,$$

and adding it all up, we have

$$[L^2, x_+] = 2\hbar x_+ L_z + 2\hbar^2 x_+ - 2\hbar z L_+ \,.$$

End Solution

(b) Show that

$$L_z x_+ |l, l\rangle = x_+ L_z |l, l\rangle + \hbar x_+ |l, l\rangle = \hbar(l + 1) x_+ |l, l\rangle \,,$$

and

$$L^2 x_+ |l, l\rangle = \hbar^2 l(l + 1) x_+ |l, l\rangle + 2\hbar^2 (l + 1) x_+ |l, l\rangle = \hbar^2 (l + 1)(l + 2) x_+ |l, l\rangle \,.$$

This means that x_+ is the ladder or raising operator for states in which $m = l$.

Solution

This is a piece of cake after part (a).

$$L_z x_+ |l,l\rangle = x_+ L_z |l,l\rangle + [L_z, x_+] |l,l\rangle = x_+ L_z |l,l\rangle + \hbar x_+ |l,l\rangle = x_+(l+1) |l,l\rangle \ .$$

Similarly,

$$\begin{aligned} L^2 x_+ |l,l\rangle &= x_+ L^2 |l,l\rangle + [L^2, x_+] |l,l\rangle \\ &= x_+ L^2 |l,l\rangle + \left(2\hbar x_+ L_z + 2\hbar^2 x_+ - 2\hbar z L_+\right) |l,l\rangle \\ &= x_+ \hbar^2 \left(l(l+1) + 2l + 2 + 0\right) |l,l\rangle \\ &= \hbar^2 (l+1)(l+2) x_+ |l,l\rangle \ . \end{aligned}$$

End Solution

So, any state $|l,m\rangle$ can be found by applying the operator x_+ to the state $|0,0\rangle$ l times and then applying L_- $l-m$ times.

$$|l,m\rangle = C L_-^{l-m} x_+^l |0,0\rangle \ ,$$

where C is a normalization constant. We can show that $|0,0\rangle$ is independent of angle. $\boldsymbol{L}|0,0\rangle = 0$, so rotating the state with $U_{\delta\varphi}$ introduced in lecture just gives back $|0,0\rangle$. This means, $|0,0\rangle$ must be a constant.

(c) Determine $|l,l\rangle$ (equivalently, $Y_{ll}(\theta,\phi)$) up to a phase using the $x_\pm$. Hint: r commutes with $\boldsymbol{L}$, L^2, and $\boldsymbol{x}$, so it is just a constant as far as all these operators are concerned.

Solution

$$|l,l\rangle = C x_+^l \cdot 1 = C(x+iy)^l \cdot 1 = C\left(r(\sin\theta\cos\phi + i\sin\theta\sin\phi)\right)^l \cdot 1 = C r^l \sin^l\theta e^{il\phi} \cdot 1 \ ,$$

where C is a normalization constant to be determined and we have replaced the constant $|0,0\rangle$ by 1 (which will be omitted in the subsequent discussion).

$$\langle l,l \mid l,l\rangle = 1 = |C|^2 r^{2l} \int_0^{2\pi} d\phi \int_0^{\pi} \sin^{2l}\theta \, \sin\theta \, d\theta = 2\pi |C|^2 r^{2l} \int_{-1}^{+1} \left(1-x^2\right)^l \, dx \ .$$

One can look up the integral or integrate by parts l times.

$$\begin{aligned}
\int_{-1}^{+1} \left(1-x^2\right)^l \, dx &= \left(1-x^2\right)^l x \Big|_{-1}^{+1} + 2l \int_{-1}^{+1} \left(1-x^2\right)^{l-1} x^2 \, dx \\
&= \frac{2l}{3} \left(1-x^2\right)^{l-1} x^3 \Big|_{-1}^{+1} + \frac{2^2 l(l-1)}{3} \int_{-1}^{+1} \left(1-x^2\right)^{l-2} x^4 \, dx \\
&= \frac{2^2 l(l-1)}{3 \cdot 5} \left(1-x^2\right)^{l-2} x^5 \Big|_{-1}^{+1} \\
&\quad + \frac{2^3 l(l-1)(l-2)}{3 \cdot 5} \int_{-1}^{+1} \left(1-x^2\right)^{l-3} x^6 \, dx \\
&\cdots \\
&= \frac{2^l l!}{3 \cdot 5 \cdots (2l-1)} \int_{-1}^{+1} x^{2l} \, dx \\
&= 2 \frac{2^l l!}{3 \cdot 5 \cdots (2l+1)} \\
&= 2 \frac{2^l l!}{(2l+1)!!} \, .
\end{aligned}$$

We deduce that, up to a phase,

$$C = \frac{1}{r^l} \sqrt{\frac{(2l+1)!!}{4\pi \, 2^l l!}} \, ,$$

and

$$|l, l\rangle = \sqrt{\frac{(2l+1)!!}{4\pi \, 2^l l!}} \sin^l \theta \, e^{il\phi} \, .$$

Note that this is almost $Y_{ll}(\theta, \phi)$. It's missing $(-1)^l$, but this phase is determined by convention!

——————— End Solution ———————

From here one could go on to use L_- to determine (up to a phase) all the angular momentum eigenfunctions (for integer l). The normalization constants were given in lecture. However, it's unlikely that this will lead to new insights, so this problem ends here!

Appendix. Since we were somewhat rushed with the coverage of $P_{lm}s$ and $Y_{lm}s$, I include a few items here. Much more can be found in any reference on mathematical functions such as Abramowitz and Stegun or any quantum text.

Associated Legendre equation. After separation of variables θ and ϕ (so the solutions for ϕ are $\exp(\pm im\phi)$) the equation for θ becomes

$$\left(\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial}{\partial\theta}\right)+l(l+1)-\frac{m^2}{\sin^2\theta}\right)f(\theta)=0\,.$$

This is the associated Legendre equation and it's customary to change the variable to $\mu=\cos\theta$.

$$\left((1-\mu^2)\frac{d^2}{d\mu^2}-2\mu\frac{d}{d\mu}+l((l+1)-\frac{m^2}{1-\mu^2}\right)P_{lm}(\mu)=0\,,$$

where the non-singular (at $\mu=\pm1$) solution has been written as $P_{lm}(\mu)$ which is known as an associated Legendre function. When $m=0$, the equation is known as the Legendre equation with regular solutions $P_{l0}(\mu)=P_l(\mu)$ which are Legendre polynomials.

$$P_l(\mu)=\frac{(-1)^l}{2^l l!}\left(\frac{d}{d\mu}\right)^l(1-\mu^2)^l\,.$$

$P_l(\mu)$ is an l^{th} order polynomial in μ and is either even or odd depending on whether l is even or odd. The normalization (be careful when consulting references, not everyone uses the same) is $P_l(1)=1$. The associated Legendre functions are given by ($m\ge0$),

$$P_{lm}(\mu)=(1-\mu^2)^{m/2}\left(\frac{d}{d\mu}\right)^m P_l(\mu)=\frac{(-1)^l}{2^l l!}(1-\mu^2)^{m/2}\left(\frac{d}{d\mu}\right)^{l+m}(1-\mu^2)^l\,.$$

For $m=l$, the associated Legendre function is particularly simple. The farthest right factor is a polynomial in l in which the highest power is $(-\mu^2)^l$ Since the polynomial is differentiated $2l$ times, only this term survives. The $(-1)^l$ cancels the $(-1)^l$ in front. The l derivatives produce a factor $2l!$ which combined with the other factors in front produces $(2l-1)\cdot(2l-3)\cdot(2l-5)\cdots3\cdot1$ which is often abbreviated $(2l-1)!!$ where !! is read "double factorial." So $P_{ll}(\mu)=(2l-1)!!(1-\mu^2)^{l/2}=(2l-1)!!\sin^l\theta$. The associated Legendre polynomials with different l, but the same $m\ge0$, are orthogonal on the interval -1 to $+1$,

$$\int_{-1}^{+1}d\mu\,P_{lm}(\mu)P_{l'm}(\mu)=\frac{2}{2l+1}\frac{(l+m)!}{(l-m)!}\delta_{ll'}\,.$$

The normalized, orthogonal, complete eigenfunctions of L^2 and L_z (for orbital angular momentum) are the spherical harmonics which are defined in terms of the associated Legendre functions for the polar angle and the azimuthal wave for the azimuthal angle. For $m\ge0$, these are

$$Y_{lm}(\theta,\phi)=(-1)^m\sqrt{\frac{2l+1}{4\pi}\frac{(l-m)!}{(l+m)!}}P_{lm}(\cos\theta)e^{im\phi}\,,$$

and for negative m, we use

$$Y_{l,-m}(\theta,\phi) = (-1)^m Y_{lm}^*(\theta,\phi)\,.$$

The ortho-normality relation is

$$\int_0^\pi \sin\theta\,d\theta \int_0^{2\pi} d\phi\,Y_{lm}(\theta,\phi)Y_{l'm'}^*(\theta,\phi) = \delta_{ll'}\delta_{mm'}\,.$$

The completeness relation is

$$\sum_{l=0}^{\infty}\sum_{m=-l}^{+l} Y_{lm}(\theta,\phi)Y_{lm}^*(\theta',\phi') = \frac{1}{\sin\theta}\,\delta(\theta-\theta')\,\delta(\phi-\phi')\,.$$

Some of the low order spherical harmonics are,

$$\begin{aligned}
Y_{00} &= +\sqrt{\frac{1}{4\pi}} \\
Y_{10} &= +\sqrt{\frac{3}{4\pi}}\,\cos\theta \\
Y_{11} &= -\sqrt{\frac{3}{8\pi}}\,\sin\theta e^{i\phi} \\
Y_{20} &= +\sqrt{\frac{5}{16\pi}}\,\left(3\cos^2\theta - 1\right) \\
Y_{21} &= -\sqrt{\frac{15}{8\pi}}\,\sin\theta\cos\theta e^{i\phi} \\
Y_{22} &= +\sqrt{\frac{15}{32\pi}}\,\sin^2\theta e^{2i\phi} \\
Y_{30} &= +\sqrt{\frac{7}{16\pi}}\,\left(5\cos^3\theta - 3\cos\theta\right) \\
Y_{31} &= -\sqrt{\frac{21}{64\pi}}\,\sin\theta\left(5\cos^2\theta - 1\right)e^{i\phi} \\
Y_{32} &= +\sqrt{\frac{105}{32\pi}}\,\sin^2\theta\cos\theta e^{2i\phi} \\
Y_{33} &= -\sqrt{\frac{35}{64\pi}}\,\sin^3\theta e^{3i\phi} \\
&\cdots
\end{aligned}$$