1. Some Preliminaries. Assume A and B are Hermitian operators.

(a) Show that $(AB)^\dagger = B^\dagger A^\dagger$.

Solution

$\int dx\, \phi^* AB\psi = \int dx\, (A^\dagger\phi)^* B\psi = \int dx\, (B^\dagger(A^\dagger\phi))^*\psi = \int dx\, (B^\dagger A^\dagger\phi)^*\psi.$

End Solution

(b) Show that $AB = [A,B]/2 + \{A,B\}/2$ where the *anticommutator* $\{A,B\} = AB + BA$. Further, show that the anticommutator is Hermitian and the commutator is anti-Hermitian (that is, $[A,B]^\dagger = -[A,B]$). We know that expectation values of Hermitian operators are real. What can you say about the expectation value of an anti-Hermitian operator?

Solution

$[A,B]/2 + \{A,B\}/2 = (AB - BA)/2 + (AB + BA)/2 = AB.$

$\{A,B\}^\dagger = (AB)^\dagger + (BA)^\dagger = B^\dagger A^\dagger + A^\dagger B^\dagger = \{A,B\}$, so the anticommutator is Hermitian.

$[A,B]^\dagger = (AB)^\dagger - (BA)^\dagger = B^\dagger A^\dagger - A^\dagger B^\dagger = -(AB - BA) = -[A,B]$, so the commutator is anti-Hermitian.

Now let A be an anti-Hermitian operator. Then $\langle A\rangle = \int dx\, \psi^*(A\psi)$. Also $\langle A^\dagger\rangle = \int dx\, (A\psi)^*\psi = -\int dx\, \psi^*(A\psi)$ which means that $\langle A\rangle$ is pure imaginary.

End Solution

2. Generalized Uncertainty Principle. Let A and B be Hermitian operators. Define the "uncertainty" in A by the square root of the mean square deviation from the mean: $\Delta A = \sqrt{\langle (A - \langle A \rangle)^2 \rangle} = \sqrt{\langle A^2 \rangle - \langle A \rangle^2}$. Show that

$$\Delta A \, \Delta B \geq \frac{1}{2} |\langle [A, B] \rangle| \, .$$

Hint: Use the Schwarz Inequality and the results from problem 1. Comment: The commutator of x and p_x is $i\hbar$, so $\Delta x \, \Delta p_x \geq \hbar/2$.

Solution

Define new Hermitian operators $A' = A - \langle A \rangle$ and $B' = B - \langle B \rangle$. Then by the Schwarz inequality,

$$\langle A'^2 \rangle \langle B'^2 \rangle \geq |\langle A' B' \rangle|^2 \, ,$$

or

$$\Delta A \, \Delta B \geq |\langle A' B' \rangle| = |\langle [A', B'] \rangle / 2 + \langle \{A', B'\} \rangle / 2| \geq |\langle [A', B'] \rangle| / 2 \, .$$

Since the expectation value of the commutator is imaginary and the anticommutator is real, each makes a positive contribution to the absolute value, and the anticommutator can be dropped without changing the inequality in the last step. So,

$$\Delta A \, \Delta B \geq |\langle [A', B'] \rangle| / 2 = |\langle [A, B] - [A, \langle B \rangle] - [\langle A \rangle, B] + [\langle A \rangle, \langle B \rangle] \rangle| / 2 = |\langle [A, B] \rangle| / 2 \, .$$

$\langle A \rangle$ and $\langle B \rangle$ are just numbers, so they commute with the operators and the commutators involving them are 0.

End Solution

3. We saw in lecture that the eigenfunction of the momentum operator with eigenvalue p is $f_p(x) = (1/\sqrt{2\pi\hbar})\exp(ipx/\hbar)$. We are working in one dimension here and we are assuming (almost) the δ-function normalization described in lecture. (The extra $\sqrt{\hbar}$ has been inserted for convenience below.) We also stated that the eigenfunctions of a Hermitian operator form a complete set. This means that for any wave function, we should be able to write

$$\psi(x) = \sum_p c_p f_p(x) \, ,$$

where c_p are the expansion coefficients. Since p is a continuous variable, the coefficients become a function and we should write:

$$\psi(x) = \int_{-\infty}^{+\infty} \varphi(p) e^{ipx/\hbar} \frac{dp}{\sqrt{2\pi\hbar}} \, ,$$

where $\varphi(p)$ plays the role of the expansion coefficients. Either $\psi(x)$ or $\varphi(p)$ is suitable for describing the state of the system. If we are using $\psi(x)$, we are using a position space or *configuration space* description. If we are using $\varphi(p)$, it's a *momentum space* description.

(a) Show how to determine $\varphi(p)$ from $\psi(x)$. (The above already gave an expression for $\psi(x)$ in terms of $\varphi(p)$.)

Solution

We take the dot product of $f_p(x)$ with $\psi(x)$.

$$\begin{aligned}
\langle f_p|\psi\rangle &= \int dx\, \frac{1}{\sqrt{2\pi\hbar}} e^{-ipx/\hbar}\psi(x) \\
&= \int dx\, \frac{1}{\sqrt{2\pi\hbar}} e^{-ipx/\hbar} \int dp'\, \varphi(p') \frac{1}{\sqrt{2\pi\hbar}} e^{ip'x/\hbar} \\
&= \int dp'\, \varphi(p') \int dx\, \frac{1}{2\pi\hbar} e^{i(p'-p)x/\hbar} \\
&= \int dp'\, \varphi(p')\delta(p'-p) \\
&= \varphi(p) \, .
\end{aligned}$$

The configuration and momentum space wavefunctions are just Fourier transforms of each other!

End Solution

(b) What is the meaning of

$$\int_{p_1}^{p_2} |\varphi(p)|^2 \, dp \, ?$$

Solution

It's the probability that a measurement of the momentum produces a result between p_1 and p_2. (Remember, the square of an expansion coefficient is the probability of being in the corresponding state.)

End Solution

(c) What are the position and momentum operators in momentum space? (You can probably make a good guess, but justify your answer!)

Solution

We know the operators in configuration space, so let's operate in configuration space and transform to momentum space.

$$\begin{aligned}
\langle f_p | x\psi \rangle &= \int dx \, \frac{1}{\sqrt{2\pi\hbar}} e^{-ipx/\hbar} x\psi(x) \\
&= \int dx \, \frac{1}{\sqrt{2\pi\hbar}} e^{-ipx/\hbar} \, x \int dp' \, \varphi(p') \frac{1}{\sqrt{2\pi\hbar}} e^{ip'x/\hbar} \\
&= \int dp' \, \varphi(p') \int dx \, x \, \frac{1}{2\pi\hbar} e^{i(p'-p)x/\hbar} \\
&= \int dp' \, \varphi(p') \int dx \, \frac{\hbar}{i} \frac{\partial}{\partial p'} \frac{1}{2\pi\hbar} e^{i(p'-p)x/\hbar} \\
&= \int dp' \, \left(-\frac{\hbar}{i} \frac{\partial}{\partial p'} \varphi(p') \right) \int dx \, \frac{1}{2\pi\hbar} e^{i(p'-p)x/\hbar} \qquad \text{integration by parts} \\
&= \int dp' \, \left(-\frac{\hbar}{i} \frac{\partial}{\partial p'} \varphi(p') \right) \delta(p'-p) \\
&= -\frac{\hbar}{i} \frac{\partial}{\partial p} \varphi(p) \, .
\end{aligned}$$

The position operator is $(-\hbar/i)(\partial/\partial p)$ as you might have guessed. By doing essentially the same thing but operating with $(\hbar/i)(\partial/\partial x)$ we would find that the momentum operator is multiplication by p, also as you might have guessed. Note that it's still the case that $[x, p] = i\hbar$.

End Solution

4. Projection operators. It's often the case that we want to find the "component" of a function "parallel" to another function. We just take the dot product with the second function, but then we also need to multiply by the second function. A handy notation is

$$|\psi\rangle\langle\psi| \,.$$

This projects onto ψ. Operating on $|\varphi\rangle$, we get

$$|\psi\rangle\langle\psi|\varphi\rangle \,,$$

which is what we want! Remember $\langle\psi|\varphi\rangle$ is just a number and $|\psi\rangle$ is the vector. Similarly, operating on $\langle\varphi|$ we get

$$\langle\varphi|\psi\rangle\langle\psi| \,,$$

which is the desired expression for the adjoint vector. Suppose you have a complete set of orthonormal basis vectors $|\psi_n\rangle$. What is a compact expression for transforming an arbitrary vector $|\varphi\rangle$ into this basis set? (This is much easier to write down than to ask!)

Solution

We want to project $|\varphi\rangle$ onto each basis vector (this gives the expansion coefficients) and then sum the coefficients times the basis vectors:

$$|\varphi\rangle = \left(\sum_n |\psi_n\rangle\langle\psi_n|\right)|\varphi\rangle \,.$$

End Solution

5. We often need to exponentiate a matrix! As an example, let A be the 2×2 matrix

$$A = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} .$$

Compute e^{itA} using two different methods:

(a) Use the Taylor expansion for the exponential:

$$e^{itA} = \sum_{n=0}^{\infty} \frac{(itA)^n}{n!} .$$

Solution

Note that

$$A^2 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = 1 .$$

So the even terms in the Taylor expansion involve the identity matrix and the odd terms involve A.

$$\begin{aligned} e^{itA} &= 1 - t^2/2! + t^4/4! - \cdots + iA(t - t^3/3! + t^5/5! - \cdots) \\ &= \cos t \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + i \sin t \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \\ &= \begin{pmatrix} \cos t & i \sin t \\ i \sin t & \cos t \end{pmatrix} . \end{aligned}$$

End Solution

(b) Use the spectral decomposition of A: write $A = \sum_{k=1}^{2} \lambda_k |\psi_k\rangle\langle\psi_k|$ with $\langle\psi_k|\psi_l\rangle = \delta_{kl}$ and use

$$e^{itA} = \sum_{k=1}^{2} e^{it\lambda_k} |\psi_k\rangle\langle\psi_k| .$$

Hint: $|\psi_k\rangle$ are just two element column vectors that are eigenvectors of the matrix A.

Solution

The first thing we need to do is find the eigenvalues and eigenfunctions. The eigenvalue equation for this matrix is

$$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} a \\ b \end{pmatrix} = \lambda \begin{pmatrix} a \\ b \end{pmatrix} .$$

or

$$\begin{pmatrix} -\lambda & 1 \\ 1 & -\lambda \end{pmatrix} \begin{pmatrix} a \\ b \end{pmatrix} = 0 .$$

This has a solution (that's not all 0s) only if the determinant vanishes. Taking the determinant and setting it to zero we get the characteristic equation

$$\lambda^2 - 1 = 0 ,$$

with solutions $\lambda = \pm 1$ and eigenfunctions (eigenvectors) belonging to $\lambda = \pm 1$, $\begin{pmatrix} 1/\sqrt{2} \\ \pm 1/\sqrt{2} \end{pmatrix}$. So

$$\begin{aligned} e^{itA} &= e^{it} \begin{pmatrix} 1/2 & 1/2 \\ 1/2 & 1/2 \end{pmatrix} + e^{-it} \begin{pmatrix} 1/2 & -1/2 \\ -1/2 & 1/2 \end{pmatrix} \\ &= \begin{pmatrix} (e^{it} + e^{-it})/2 & (e^{it} - e^{-it})/2 \\ (e^{it} - e^{-it})/2 & (e^{it} + e^{-it})/2 \end{pmatrix} \\ &= \begin{pmatrix} \cos t & i \sin t \\ i \sin t & \cos t \end{pmatrix} . \end{aligned}$$

End Solution