PHYSICS DEPARTMENT, PRINCETON UNIVERSITY

PHYSICS 505 MIDTERM EXAMINATION

October 25, 2012, 11:00am–12:20pm, Jadwin Hall A06

SOLUTIONS

This exam contains two problems. Work both problems. They count equally although they may not be the same difficulty.

Do all the work you want graded in the separate exam books.

The exam is closed everything: no books, no notes, no calculators, no computers, no cell phones, no ipods, etc.

Write legibly. If I can't read it, it doesn't count!

Put your name on all exam books that you hand in. (Only one should be necessary!!!) On the first exam book, rewrite and sign the honor pledge: *I pledge my honor that I have not violated the Honor Code during this examination.*

1. **Double Delta Functions.** A particle of mass m, moving in one dimension, is confined by a double delta function potential,

$$V(x) = -\frac{\hbar^2 b}{m}\left(\delta(x+a) + \delta(x-a)\right) ,$$

where $b > 0$. The energy of the particle is $E < 0$ with $-E = (\hbar^2\kappa^2/2m)$.

(a) Since the potential has even parity, bound states have even or odd parity. What are the general even parity ($\psi(x) = +\psi(-x)$) and odd parity ($\psi(x) = -\psi(-x)$) bound state wave functions? (Don't worry about normalization.) What are the matching conditions at $x \pm a$ for wave functions of each parity? (Note, you will solve the conditions later, here the question is "what are the conditions?")

Solution

Based on a problem from the May, 2008 Preliminary Examination.

Except at the δ-functions, the Schroedinger equation is $\psi'' = \kappa^2\psi$. Thus the wave function is $\exp(\pm\kappa x)$. For $x \to \pm\infty$, the wave function must remain finite which means we must use $\exp(-\kappa x)$ for $x > +a$ and $\pm\exp(+\kappa x)$ for $x < -a$. The upper sign refers to the even parity solutions and the lower sign to the odd parity solutions. Between the δ-functions, the even and odd parity solutions are $A\cosh\kappa x$ and $A\sinh\kappa x$. The constant A must be chosen to match the solutions at the δ-functions and there is an overall constant (which we're not interested in) to normalize the wave function.

The matching conditions are that the wave function must be continuous at $x = \pm a$ or

$$\psi_+(a) = \psi_-(a) ,$$

where $\psi_\pm(a)$ is the limit of the wave function at a as approached from the positive or negative side of a. (If the matching conditions are satisfied at $+a$ they are automatically satisfied at $-a$ since the wave functions have definite parity.) Also, the wave function must have a discontinuity in slope given by the integral over the δ-function as follows,

$$\begin{aligned}
-\frac{\hbar^2}{2m}\frac{d^2}{dx^2}\psi(x) + V(x)\psi(k) &= E\psi(x) \\
\int_{a-\epsilon}^{a+\epsilon} \frac{\hbar^2}{2m}\frac{d^2}{dx^2}\psi(x)\,dx &= \int_{a-\epsilon}^{a+\epsilon} -\frac{\hbar^2 b}{m}\delta(x-a)\psi(x)\,dx \\
\frac{1}{2}\left(\psi'_+(a) - \psi'_-(a)\right) &= -\frac{b}{2}\left(\psi_+(a) + \psi_-(a)\right) .
\end{aligned}$$

End Solution

(b) For even parity wave functions, show that there is exactly one bound state and obtain an equation whose solution determines κ in terms of b and a.

Solution

With $\psi_-(x) = A\cosh\kappa x$ and $\psi_+(x) = e^{-\kappa x}$ the continuity condition becomes

$$A\cosh\kappa a = e^{-\kappa a} ,$$

and the step in slope condition becomes

$$-\kappa e^{-\kappa a} - \kappa A\sinh\kappa a = -be^{-\kappa a} - bA\cosh\kappa a .$$

Use the continuity condition to eliminate $\exp(-\kappa a)$from the second equation,

$$-\kappa\cosh\kappa a - \kappa\sinh\kappa a = -2b\cosh\kappa a ,$$

or

$$\kappa a\,(\tanh\kappa a + 1) = 2ba .$$

Given b and a, this equation determines κ for a bound state. The right hand side of this equation is a positive number. The left hand side starts at 0 when $\kappa a = 0$ and rises monotonically to ∞ as $\kappa a \to \infty$. (It approaches the line $2\kappa a$.) Thus there is exactly one place where the left hand side crosses the right hand side which means there is exactly one solution (value of κ and hence $-E$) for given b and a.

End Solution

(c) For odd parity wave functions, what is the condition on b and a such that a bound state exists? Obtain an equation whose solution determines κ when the bound state exists.

Solution

The two matching equations are

$$A\sinh\kappa a = e^{-\kappa a} ,$$

and

$$-\kappa e^{-\kappa a} - \kappa A\cosh\kappa a = -be^{-\kappa a} - bA\sinh\kappa a .$$

Use the first to eliminate $\exp(-\kappa a)$ from the second,

$$-\kappa\sinh\kappa a - \kappa\cosh\kappa a = -2b\sinh\kappa a ,$$

or

$$\kappa a\,(\coth\kappa a + 1) = 2ba .$$

Given b and a, this equation determines κ for a bound state. Again, the right hand side of this equation is a positive number. However, the left hand side starts at 1 (not 0) when $\kappa a \to 0$ and rises monotonically to ∞ as $\kappa a \to \infty$. (It approaches the line $2\kappa a$.) There is no solution (in real numbers!) when $2ba < 1$. The condition for a bound state is $2ba \geq 1$.

——— End Solution ———

2. **Well!** Consider an infinitely deep (one-dimensional) potential well, $V(x) = 0$ for $0 \le x \le 2a$ and $V(x) = \infty$ for $x < 0$ or $x > 2a$. A particle of mass m moves in this potential well.

(a) The energy eigenvalues and time independent eigenfunctions can be labeled by an integer $n, n = 1, 2, 3, \ldots$, where the energy increases as n increases. What are the eigenvalues, E_n, and the corresponding (normalized) eigenfunctions, $\psi_n(x)$? What is the interpretation of n?

——— Solution ———

The time independent Schroedinger equation within the well is just

$$-\frac{\hbar^2}{2m}\frac{d^2}{dx^2}\psi_n = E_n\psi_n \; .$$

This means the solutions are $e^{\pm ik_n x}$, where $k_n = \sqrt{2mE/\hbar^2}$. The eigenfunction must be zero at the boundaries and beyond since the potential is infinite there. We take $\psi_n(x) = \sin(k_n x)$ in order to satisfy the boundary condition at $x = 0$. The boundary condition at $x = 2a$ is satisfied if $k_n = n\pi/2a$ where n is any integer, $n = 1, 2, 3, \ldots$. n is the number of half wavelengths that fit across the well and $n - 1$ is the number of nodes in the well. To normalize the eigenfunction, we require a normalization constant $1/\sqrt{a}$. To summarize, for integers $n \ge 1$, the energies and eigenfunctions are

$$k_n = \frac{n\pi}{2a} \; , \quad E_n = \frac{n^2\pi^2\hbar^2}{8ma^2} \; , \quad \psi_n(x) = \frac{1}{\sqrt{a}}\sin k_n x \; , 0 \le x \le 2a \; , \psi_n(x) = 0 \; , \text{ otherwise} \; .$$

——— End Solution ———

(b) At $t = 0$, the wave function of the particle in the well is

$$\psi(x, t = 0) = \sqrt{\frac{2}{a}}\sin\frac{\pi x}{a} \; , \text{ for } 0 \le x \le a \; , \quad \text{and } \psi(x, t = 0) = 0 \; , \text{ otherwise.}$$

That is, the particle is in the left half of the well. (This state might arise, for example, if the particle were in the ground state of a well of width a and the width of the well were suddenly doubled to $2a$ by moving the right boundary.) What is the wave function for $t > 0$? Also, if a measurement of the energy of the particle is made after $t = 0$, what is the probability of obtaining E_n, where n is one of the energy eigenvalues you obtained in part (a)? Hint: you really should know your trig identities, but in case you don't remember,

$$\sin A \sin B = (\cos(A - B) - \cos(A + B))/2 \; .$$

Solution

We need to expand the wave function at $t = 0$ in the eigenfunctions we found in part (a):

$$\psi(x,0) = \sum_{n=1}^{\infty} c_n \psi_n(x) = \sum_{n=1}^{\infty} c_n \frac{1}{\sqrt{a}} \sin\frac{n\pi x}{2a} \, .$$

If we multiply both sides by $\psi_m^*(x)$ and integrate from 0 to $2a$, we get

$$\int_0^{2a} \psi_m^*(x)\psi(x,0)\, dx = c_m \, ,$$

or

$$c_m = \int_0^a \sqrt{\frac{1}{a}} \sin\frac{m\pi x}{2a} \sqrt{\frac{2}{a}} \sin\frac{\pi x}{a}\, dx \, ,$$

where the upper limit of integration has been set to a since ψ is 0 for $x > a$. We change variables to $y = \pi x/2a$ and

$$\begin{aligned}
c_m &= \frac{2\sqrt{2}}{\pi}\int_0^{\pi/2} \sin my \, \sin 2y \, dy \\
&= \frac{\sqrt{2}}{\pi}\int_0^{\pi/2} (\cos(m-2)y - \cos(m+2)y)\, dy \\
&= \frac{\sqrt{2}}{\pi}\left(\frac{1}{m-2}\sin(m-2)\frac{\pi}{2} - \frac{1}{m+2}\sin(m+2)\frac{\pi}{2}\right) \quad m = 2 \text{ is a special case.} \\
&= \frac{\sqrt{2}}{\pi}\left(\frac{1}{m-2}(-1)^{(m+1)/2} - \frac{1}{m+2}(-1)^{(m+1)/2}\right) \quad m \text{ odd, 0 if } m > 2 \text{ and even} \\
&= \frac{4\sqrt{2}}{\pi(m^2-4)}(-1)^{(m+1)/2} \quad m \text{ odd} \, .
\end{aligned}$$

The special case $m = 2$ is

$$c_2 = \frac{2\sqrt{2}}{\pi}\int_0^{\pi/2} \sin^2 2y \, dy = \frac{1}{\sqrt{2}} \, ,$$

and all the other even m c_m are zero. Now that we know the expansion coefficients we just add $e^{-i\omega_n t}$ to each term in the sum. $\omega_n = E_n/\hbar = n^2\hbar\pi^2/8ma^2 = n^2\omega_1$,

$$\psi(x,t) = \frac{1}{\sqrt{2}}\frac{1}{\sqrt{a}} \sin\frac{2\pi x}{2a} e^{-i4\omega_1 t} + \sum_{\substack{n=1 \\ n \text{ odd}}}^{\infty} \frac{4\sqrt{2}}{\pi(n^2-4)}(-1)^{(n+1)/2} \frac{1}{\sqrt{a}} \sin\frac{n\pi x}{2a} e^{-in^2\omega_1 t} \, .$$

Also, now that we have the expansion coefficients, the probability of obaining E_n in a measurement of the energy is just $|c_n|^2$. So

$$P_2 = \frac{1}{2} \, , \quad P_n = \frac{32}{\pi^2(n^2-4)^2} \quad n \text{ odd} \, , \quad P_n = 0 \quad n \text{ even, } n > 2 \, .$$

Random interesting observation: since the probabilities must sum to 1, we have just shown that

$$\pi^2 = \sum_{\substack{n=1 \\ n \text{ odd}}}^{\infty} \frac{64}{(n^2-4)^2} \, .$$

End Solution

(c) The probability density oscillates with period T. What is T? Hint: you can solve this part without having done part (b).

Solution

The probability density is $\psi^*(x,t)\psi(x,t)$ and will include terms with factors like

$$e^{im^2\omega_1 t - in^2\omega_1 t} \, ,$$

plus the complex conjugate. Here, $\omega_1 = \hbar\pi^2/8ma^2$ and m and n are the indices of two of the eigenfunctions included in sum. A little thought will show that the only frequency that divides all possible frequencies in the sum is ω_1. Since all the frequencies are multiples of ω_1, the wave function will repeat with period $T = 2\pi/\omega_1 = 16ma^2/\pi\hbar$.

End Solution