PHYSICS DEPARTMENT, PRINCETON UNIVERSITY

PHYSICS 505 MIDTERM EXAMINATION

October 27, 2011, 11:00am–12:20pm, Jadwin Hall A06

SOLUTIONS

This exam contains two problems. Work both problems. They count equally although they may not be the same difficulty.

Do all the work you want graded in the separate exam books.

The exam is closed everything: no books, no notes, no calculators, no computers, no cell phones, no ipods, etc.

Write legibly. If I can't read it, it doesn't count!

Put your name on all exam books that you hand in. (Only one should be necessary!!!) On the first exam book, rewrite and sign the honor pledge: *I pledge my honor that I have not violated the Honor Code during this examination.*

1. Particles of mass m move in one dimension (the x direction) with energy E and wave number $\pm k$ with $E = \hbar^2 k^2/2m$. They move under the influence of a potential which consists of δ-functions at $x = \pm a$.

$$V(x) = \frac{\hbar^2 b}{m}\delta(x+a) + \frac{\hbar^2 b}{m}\delta(x-a)\,.$$

The wave function for $x < -a$ can be written as $Ae^{+ikx} + Be^{-ikx}$ and the wave function for $x > +a$ is $Ce^{+ikx} + De^{-ikx}$.

(a) The relation between the coefficients A and B and the coefficients C and D can be written in matrix form as

$$\begin{pmatrix} A \\ B \end{pmatrix} = \begin{pmatrix} M_{11} & M_{12} \\ M_{21} & M_{22} \end{pmatrix} \begin{pmatrix} C \\ D \end{pmatrix}\,.$$

An incident wave comes from $x = -\infty$. In terms of the matrix components above, what are the reflection and transmission coefficients?

Solution

For an incident wave from the left, we must have $D = 0$. Then $A = M_{11}C$ and $B = M_{21}C$. The transmission coefficient is

$$t = \left|\frac{C}{A}\right|^2 = \frac{1}{|M_{11}|^2}$$

and the reflection coefficient is

$$r = \left|\frac{B}{A}\right|^2 = \frac{|M_{21}|^2}{|M_{11}|^2}\,.$$

End Solution

(b) Determine the matrix introduced in part (a).

Solution

For $-a < x < +a$, take $\psi = Ee^{+ikx} + Fe^{-ikx}$. (Here, E is not the energy!)

We need to match the waves at $x = -a$. Continuity of ψ gives

$$Ae^{-ika} + Be^{+ika} = Ee^{-ika} + Fe^{+ika}\,,$$

Integrating Schroedinger's equation from $-a-\epsilon$ to $-a+\epsilon$ gives an expression for the step in ψ' at $x = -a$.

$$\frac{-\hbar^2}{2m}\int_{-a-\epsilon}^{-a+\epsilon} dx\,\psi'' + \int_{-a-\epsilon}^{-a+\epsilon} dx\,V\psi = \int_{-a-\epsilon}^{-a+\epsilon} dx\,\frac{\hbar^2 k^2}{2m}\psi$$

$$\frac{-\hbar^2}{2m}\left(ikEe^{-ika} - ikFe^{+ika} - ikAe^{-ika} + ikBe^{+ika}\right) + \frac{\hbar^2 b}{2m}\left(Ae^{-ika} + Be^{+ika} + Ee^{-ika} + Fe^{+ika}\right) = 0\,,$$

where we have taken the average of the wave functions for $x < -a$ and $x > -a$. Rearranging, we get

$$(b + ik)Ae^{-ika} + (b - ik)Be^{+ika} = (-b + ik)Ee^{-ika} + (-b - ik)Fe^{+ika} .$$

We summarize our two equations in a matrix equation

$$\begin{pmatrix} e^{-ika} & e^{+ika} \\ (b+ik)e^{-ika} & (b-ik)e^{+ika} \end{pmatrix} \begin{pmatrix} A \\ B \end{pmatrix} = \begin{pmatrix} e^{-ika} & e^{+ika} \\ (-b+ik)e^{-ika} & (-b-ik)e^{+ika} \end{pmatrix} \begin{pmatrix} E \\ F \end{pmatrix} .$$

Inverting the first matrix and multiplying both sides by the inverse, we get

$$\begin{pmatrix} A \\ B \end{pmatrix} = \begin{pmatrix} 1 + ib/k & (+ib/k)e^{+2ika} \\ (-ib/k)e^{-2ika} & 1 - ib/k \end{pmatrix} \begin{pmatrix} E \\ F \end{pmatrix} .$$

The matrix relating E and F to C and D is the same except the sign of a switches. Multiplying the two together we get

$$\begin{pmatrix} M_{11} & M_{12} \\ M_{21} & M_{22} \end{pmatrix} =$$
$$\begin{pmatrix} \left(1 + i\frac{b}{k}\right)^2 + \left(\frac{b}{k}\right)^2 e^{4ika} & i\frac{b}{k}\left(1 + i\frac{b}{k}\right)e^{-2ika} + i\frac{b}{k}\left(1 - i\frac{b}{k}\right)e^{2ika} \\ -i\frac{b}{k}\left(1 + i\frac{b}{k}\right)e^{-2ika} - i\frac{b}{k}\left(1 - i\frac{b}{k}\right)e^{2ika} & \left(1 - i\frac{b}{k}\right)^2 + \left(\frac{b}{k}\right)^2 e^{-4ika} \end{pmatrix}$$

End Solution

(c) In terms of k, a, and b, what is the condition for 100% transmission of the incident wave. You may leave your answer as a transcendental equation, but it should be simplified as much as possible.

Solution

If the wave is completely transmitted, there is no reflected wave, which means $M_{21} = 0$:

$$\begin{aligned} 0 &= -i\frac{b}{k}\left(e^{-2ika} + e^{2ika}\right) + \left(\frac{b}{k}\right)^2 \left(e^{-2ika} - e^{2ika}\right) \\ ka &= -ba \tan 2ka . \end{aligned}$$

Many of you wrote $M_{11} = 1$. This is wrong. The condition is $|M_{11}| = 1$ or $|M_{11}|^2 = 1$. It's much easier and cleaner to use $M_{21} = 0$. If you use this condition to eliminate $\exp(4ika)$ from M_{11}, you find that at 100% transmission,

$$M_{11} = \frac{1 + ib/k}{1 - ib/k} \neq 1 .$$

End Solution

2. A one dimensional particle is confined to a box between $x = -a$ and $x = +a$ by the potential $V(x) = 0$ for $|x| < a$ and $V = \infty$ for $|x| > a$. The mass of the particle is m. The wave function at $t = 0$ is 0 for $|x| > a$ (of course!) and for $|x| < a$ it is

$$\psi(x,0) = \frac{1}{\sqrt{6a}}\left(\sqrt{3}\cos\frac{\pi x}{2a} + \sqrt{2}\sin\frac{\pi x}{a} + \cos\frac{3\pi x}{2a}\right) .$$

(a) What is the wave function at time t?

Solution

The wave function is the sum of three energy eigenfunctions, so all we need do is insert the appropriate $\exp(-i\omega t)$ for each eigenfunction. The eigenfunctions are of the form $\cos(kx)$ (even functions) and $\sin(kx)$ (odd functions) with $k = n\pi/2a$ where $n = 1, 3, 5, \ldots$ for the even functions and $n = 2, 4, 6, \ldots$ for the odd functions. The energies are $\hbar^2k^2/2m$, so $\omega_n = n^2\hbar\pi^2/8ma^2$ and

$$\psi(x,t) = \frac{1}{\sqrt{6a}}\left(\sqrt{3}\cos\frac{\pi x}{2a}\,e^{-i\dfrac{\hbar\pi^2 t}{8ma^2}} + \sqrt{2}\sin\frac{\pi x}{a}\,e^{-i\dfrac{4\hbar\pi^2 t}{8ma^2}} + \cos\frac{3\pi x}{2a}\,e^{-i\dfrac{9\hbar\pi^2 t}{8ma^2}}\right) .$$

End Solution

(b) A measurement of the energy is made. What are the possible values of the energy that can result when the wave function is ψ? What is the probability of obtaining each energy?

Solution

A measurement of the energy must result in one of the energies corresponding to one of the eigenfunctions in the sum for ψ. That is, one of the three energies found in part (a). Inspection shows that ψ is properly normalized and is of the form

$$\psi(x,t) = \sum_n c_n\varphi_n(x)e^{-i\omega_n t} ,$$

where φ_n is a normalized eigenfunction belonging to energy E_n. The c_n are $1/\sqrt{2}$, $1/\sqrt{3}$, and $1/\sqrt{6}$, so the energies and probabilities are:

$$E_1 = \frac{\hbar^2\pi^2}{8ma^2} , \qquad P_1 = \frac{1}{2}$$
$$E_2 = \frac{4\hbar^2\pi^2}{8ma^2} , \qquad P_2 = \frac{1}{3}$$
$$E_3 = \frac{9\hbar^2\pi^2}{8ma^2} , \qquad P_3 = \frac{1}{6}$$

End Solution