1. Spinor rotations. Somewhat based on a problem in Schwabl.

(a) Suppose $\boldsymbol{n}$ is a unit vector. We are interested in $\boldsymbol{n} \cdot \boldsymbol{\sigma}$. Show that $(\boldsymbol{n} \cdot \boldsymbol{\sigma})^2 = I$, where I is the identity matrix.

The unitary operator which generates a rotation by φ about the axis $\boldsymbol{n}$ in spinor space is

$$U = e^{i\varphi \boldsymbol{n} \cdot \boldsymbol{S}/\hbar} .$$

(b) Show that $U = \cos\varphi/2 + i\boldsymbol{n} \cdot \boldsymbol{\sigma} \sin\varphi/2$.

(c) Show that

$$U\boldsymbol{\sigma}U^\dagger = \boldsymbol{n}(\boldsymbol{n} \cdot \boldsymbol{\sigma}) - \boldsymbol{n} \times (\boldsymbol{n} \times \boldsymbol{\sigma}) \cos\varphi + (\boldsymbol{n} \times \boldsymbol{\sigma}) \sin\varphi .$$

You might find it helpful to remember that a product of two Pauli matrices gives the identity matrix or plus or minus i times the third Pauli matrix according to the formula

$$\sigma_i \sigma_j = \delta_{ij} + i\epsilon_{ijk}\sigma_k ,$$

where the summation convention is assumed and where we have just used 1 in place of I.

(d) Consider the special case $\boldsymbol{n} = \boldsymbol{e}_z$ and discuss infinitesimal rotations.

(e) For a spinor

$$\chi = \begin{pmatrix} \alpha_+ \\ \alpha_- \end{pmatrix} ,$$

calculate the transformed spinor χ' for $\boldsymbol{n} = \boldsymbol{e}_z$.

(f) What happens to the spinor when $\varphi = 2\pi$?

2. Spin precession.

(a) The spinors

$$|{\uparrow}\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \qquad |{\downarrow}\rangle = \begin{pmatrix} 0 \\ 1 \end{pmatrix} ,$$

are the eigenfunctions of σ_z. They are an orthonormal, complete basis in which σ_z is diagonal. In this same basis, what are the eigenfunctions (spinors) of σ_x and σ_y?

Now we consider a spin 1/2 particle in a uniform magnetic field. For definiteness, we suppose it's an electron, so the Hamiltonian for the spin degree of freedom is

$$H = \frac{ge}{2mc}\boldsymbol{S} \cdot \boldsymbol{B} \approx \frac{e}{mc}\boldsymbol{S} \cdot \boldsymbol{B} .$$

(b) The state of the system is represented by the spinor $\chi = \begin{pmatrix} \chi_+ \\ \chi_- \end{pmatrix}$. What is the equation of motion for χ? After writing down the equation for the general case, specialize to the case $\boldsymbol{B} = B\boldsymbol{e}_z$. Your results should depend on the frequency $\omega = eB/mc$ which will turn out to be the precession frequency.

(c) In the case with $\boldsymbol{B} = B\boldsymbol{e}_z$, and in the Heisenberg representation, what is the time dependence of the spin operator, $\boldsymbol{S}(t)$?

(d) In the case with $\boldsymbol{B} = B\boldsymbol{e}_z$, and in the Schroedinger representation, what is the time dependence of the spinor, $\chi(t)$?

(e) At $t = 0$, the spin is aligned along the x axis. What is the probability of getting $\hbar/2$ in a measurement of S_z at time t.

(f) At $t = 0$, the spin is aligned along the x axis. What is the probability of getting $\hbar/2$ in a measurement of S_x at time t.

3. Magnetic Resonance. This problem continues from where problem 2 ended. We have an electron in a uniform magnetic field along the z-axis. In addition, we have a small, time variable field along the x-axis: $B_x = B_p \cos \omega t$, where ω is determined by the field in the z-direction, $\omega = eB/mc$. We suppose that at $t = 0$, the electron has spin down along the z-axis. The experimental picture is that the magnetic moment of the electron is aligned with the field (so the spin is anti-aligned) and the "probe" field is turned on at $t = 0$. What happens? In particular what is $\langle S_z(t) \rangle$? Use the interaction representation to solve this problem.

Hint: The probe field is oscillating along the x-axis. However you can think of it as the sum of two fields, each rotating in the xy-plane with angular velocity ω. One rotates in the positive direction and one rotates in the negative direction. When you transform the Hamiltonian due to the probe field to the interaction representation, one of these will wind up stationary, and the other will wind up rotating at 2ω. Make an argument for why the 2ω term can be ignored and then proceed with the stationary term.

4. Combining angular momentum. Two electron spins, $\boldsymbol{S}_1$ and $\boldsymbol{S}_2$ can be summed to produce a total angular momentum $\boldsymbol{J} = \boldsymbol{S}_1 + \boldsymbol{S}_2$. The states $|j\, m\, s_1\, s_2\rangle$ can be expanded in the states $|s_1\, m_1\rangle\, |s_2\, m_2\rangle$. Determine the expansion.