1. More on Bloch Functions. We showed in lecture that the wave function for the time independent Schroedinger equation with a periodic potential could be written as a Bloch function

$$e^{iqx}u_q(x) ,$$

where u_q is periodic with the same period as the potential.

(a) Show that u_q satisfies the differential equation

$$-\frac{\hbar^2}{2m}\frac{d^2u_q}{dx^2} - \frac{i\hbar^2 q}{m}\frac{du_q}{dx} + \frac{\hbar^2 q^2}{2m}u_q + Vu_q = Eu_q ,$$

where E is the energy.

(b) In fact, any wave 1D wave function can be written in the above form (with u_q not necessarily periodic) and will satisfy the above equation. The key point is that u_q is periodic with the same period ℓ as the potential. This means we only need to solve for u_q in an interval of length ℓ, $[0, \ell]$ or $[-a, a]$, say. $(2a = \ell.)$ What are the boundary conditions on u_q at the ends of the interval? You should consider the case when V is well behaved at the ends of the interval.

(c) Solve u_q for the case of a periodic potential containing δ-function potential wells every $2a$ (the same case we did in lecture).

$$V(x) = -\lambda \sum_{n=-\infty}^{+\infty} \delta(x - a - n\ell) .$$

Since the u_q must be periodic, it doesn't matter where you place the interval of length ℓ. You might put it at $[0, \ell]$ so the potential well is in the middle of the interval.

2. The square well and the δ-function well. In lecture we discussed the energy levels in a square well $V = -V_0$ for $|x| < a$ and $V = 0$ for $|x| > a$. We also discussed the bound state of the δ-function potential well, $V = -\lambda\delta(x)$. Show that in the limit $V_0 \longrightarrow \infty$, $a \longrightarrow 0$, such that $2V_0 a \longrightarrow \lambda$, the ground state energy and the ground state wave function of the square well become the same as those for the δ-function well.

3. The variational method. This is a subject I touched on in lecture but decided to save all the fun for the homework! Suppose you have a potential for which a ground state exists (the minimum energy of a stationary state is finite). You want to know the ground state energy but the potential is sufficiently complicated that you are unable to get an exact solution for the ground state wave function or the eigenvalue problem. The variational method consists in picking a "random" function which has at least one adjustable parameter, calculating the expectation value of the energy assuming the function you picked is the wave function of the system and then varying the parameter(s) to find the minimum energy. Of course,

the function you pick must be square integrable (normalizable). Show that the minimum energy estimated this way has a lower bound which is the ground state energy. You may assume the spectrum is discrete.

4. Variational method example. Use the variational method to estimate the ground state energy of the harmonic oscillator (with Hamiltonian $H = p^2/2m + m\omega^2 x^2/2$). We already know that the ground state wave function is a Gaussian, so **don't use a Gaussian trial wave function!** (That would be "cheating.") Pick some other simple function and see how close to $\hbar\omega/2$ you get. Note that if you pick a function with discontinuities (in the function or its derivative), you need to treat the discontinuities properly.

For additional fun (not to be turned in). What wold you do if you wanted to estimate E_1 as well as E_0 using the variational method?

5. "Bound states" in a periodic potential. In lecture we discussed the one-dimensional periodic potential made from delta functions,

$$V(x) = -\lambda \sum_{n=-\infty}^{+\infty} \delta(x - a - nl) \, ,$$

where $l = 2a$, so there are δ-functions at $(2n + 1)a$ where n is any integer. In lecture we looked for energy eigenfunctions with energy $E > 0$. In this problem you are asked to find the energy eigenvalues with energy $E < 0$. These correspond to the bound state of the single delta function potential.