1. Some Preliminaries. Assume A and B are Hermitian operators.

(a) Show that $(AB)^\dagger = B^\dagger A^\dagger$.

(b) Show that $AB = [A, B]/2 + \{A, B\}/2$ where the *anticommutator* $\{A, B\} = AB + BA$. Further, show that the anticommutator is Hermitian and the commutator is anti-Hermitian (that is, $[A, B]^\dagger = -[A, B]$). We know that expectation values of Hermitian operators are real. What can you say about the expectation value of an anti-Hermitian operator?

2. Generalized Uncertainty Principle. Let A and B be Hermitian operators. Define the "uncertainty" in A by the square root of the mean square deviation from the mean: $\Delta A = \sqrt{\langle (A - \langle A \rangle)^2 \rangle} = \sqrt{\langle A^2 \rangle - \langle A \rangle^2}$. Show that

$$\Delta A \, \Delta B \geq \frac{1}{2} |\langle [A, B] \rangle| \, .$$

Hint: Use the Schwarz Inequality and the results from problem 1. Comment: The commutator of x and p_x is $i\hbar$, so $\Delta x \, \Delta p_x \geq \hbar/2$.

3. We saw in lecture that the eigenfunction of the momentum operator with eigenvalue p is $f_p(x) = (1/\sqrt{2\pi\hbar})\exp(ipx/\hbar)$. We are working in one dimension here and we are assuming (almost) the δ-function normalization described in lecture. (The extra $\sqrt{\hbar}$ has been inserted for convenience below.) We also stated that the eigenfunctions of a Hermitian operator form a complete set. This means that for any wave function, we should be able to write

$$\psi(x) = \sum_p c_p f_p(x) \,,$$

where c_p are the expansion coefficients. Since p is a continuous variable, the coefficients become a function and we should write:

$$\psi(x) = \int_{-\infty}^{+\infty} \varphi(p) e^{ipx/\hbar} \, \frac{dp}{\sqrt{2\pi\hbar}} \,,$$

where $\varphi(p)$ plays the role of the expansion coefficients. Either $\psi(x)$ or $\varphi(p)$ is suitable for describing the state of the system. If we are using $\psi(x)$, we are using a position space or *configuration space* description. If we are using $\varphi(p)$, it's a *momentum space* description.

(a) Show how to determine $\varphi(p)$ from $\psi(x)$. (The above already gave an expression for $\psi(x)$ in terms of $\varphi(p)$.)

(b) What is the meaning of

$$\int_{p_1}^{p_2} |\varphi(p)|^2 \, dp \,?$$

(c) What are the position and momentum operators in momentum space? (You can probably make a good guess, but justify your answer!)

4. Projection operators. It's often the case that we want to find the "component" of a function "parallel" to another function. We just take the dot product with the second function, but then we also need to multiply by the second function. A handy notation is

$$|\psi\rangle\langle\psi| \,.$$

This projects onto ψ. Operating on $|\varphi\rangle$, we get

$$|\psi\rangle\langle\psi|\varphi\rangle \,,$$

which is what we want! Remember $\langle\psi|\varphi\rangle$ is just a number and $|\psi\rangle$ is the vector. Similarly, operating on $\langle\varphi|$ we get

$$\langle\varphi|\psi\rangle\langle\psi| \,,$$

which is the desired expression for the adjoint vector. Suppose you have a complete set of orthonormal basis vectors $|\psi_n\rangle$. What is a compact expression for transforming an arbitrary vector $|\varphi\rangle$ into this basis set? (This is much easier to write down than to ask!)

5. We often need to exponentiate a matrix! As an example, let A be the 2×2 matrix

$$A = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} .$$

Compute e^{itA} using two different methods:

(a) Use the Taylor expansion for the exponential:

$$e^{itA} = \sum_{n=0}^{\infty} \frac{(itA)^n}{n!} .$$

(b) Use the spectral decomposition of A: write $A = \sum_{k=1}^{2} \lambda_k |\psi_k\rangle\langle\psi_k|$ with $\langle\psi_k|\psi_l\rangle = \delta_{kl}$ and use

$$e^{itA} = \sum_{k=1}^{2} e^{it\lambda_k} |\psi_k\rangle\langle\psi_k| .$$

Hint: $|\psi_k\rangle$ are just two element column vectors that are eigenvectors of the matrix A.