PHYSICS DEPARTMENT, PRINCETON UNIVERSITY

PHYSICS 505 FINAL EXAMINATION

January 18, 2012, 1:30–4:30pm, A06 Jadwin Hall

SOLUTIONS

This exam contains five problems. Work any three of the five problems. All problems count equally although some are harder than others. Do all the work you want graded in the separate exam books. *Indicate clearly which three problems you have worked and want graded.* I will only grade three problems. If you hand in more than three problems without indicating which three are to be graded, I will grade the first three, only!

The exam is closed everything: no books, no notes, no calculators, no computers, no cell phones, no ipods, etc.

Write legibly. If I can't read it, it doesn't count!

Put your name on all exam books that you hand in. (Only one should be necessary!!!) On the first exam book, rewrite and sign the honor pledge: *I pledge my honor that I have not violated the Honor Code during this examination.*

If you finish early, do not leave your exam books in the room. Instead, take them to Jessica Heslin in room 210 Jadwin Hall.

1. Quantum Zeno. In this problem we explore the "Quantum Zeno Effect." Zeno's arrow paradox is basically that when one observes an arrow in flight, it is at a particular spot at a particular instant and hence can't be moving. Not sure about Zeno, but in quantum mechanics, sufficiently rapid observations of a system can keep it from changing its state!

An electron spin interacts with a constant magnetic field. The Hamiltonian is

$$H = \mu_B \boldsymbol{B} \cdot \boldsymbol{\sigma} \; ,$$

where the g-factor of the electron has been taken to be 2, μ_B is the Bohr magneton, and $\boldsymbol{\sigma}$ is the vector of Pauli spin matrices:

$$\sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \; , \qquad \sigma_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \; , \qquad \sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \; .$$

The magnetic field is in the x-direction and has magnitude B, so the Hamiltonian becomes

$$H = \hbar\omega \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \; ,$$

where $\hbar\omega = \mu_B B$. $|{\uparrow}\rangle$ is the state with spin pointing in the $+z$-direction and $|{\downarrow}\rangle$ is the state with spin pointing in the $-z$-direction.

This problem is concerned with idealized measurements which determine the state of the spin along the z-axis. The measurements take a very short time to perform and if the state is found to be $|{\uparrow}\rangle$, the electron is left in $|{\uparrow}\rangle$ immediately after the measurement. Similarly, if the state is found to be $|{\downarrow}\rangle$, the electron is left in $|{\downarrow}\rangle$ immediately after the measurement.

The electron is prepared in state $|{\uparrow}\rangle$ at $t = 0$.

(a) What is the probability of finding the electron spin to have flipped to the state $|{\downarrow}\rangle$ at time $t_1 > 0$?

——————— Solution ———————

The stationary states of the Hamiltonian are the states $|\pm\rangle$ with spin aligned along the $\pm x$-axis. State $|\pm\rangle$ has energy $\pm\hbar\omega$. These states can be written as a superposition of $|{\uparrow}\rangle$ and $|{\downarrow}\rangle$:

$$|\pm\rangle = \frac{1}{\sqrt{2}} \left(|{\uparrow}\rangle \pm |{\downarrow}\rangle \right) \; .$$

Inverting this relation, we find

$$|{\uparrow}\rangle = \frac{1}{\sqrt{2}} \left(|+\rangle + |-\rangle \right) \; , \qquad |{\downarrow}\rangle = \frac{1}{\sqrt{2}} \left(|+\rangle - |-\rangle \right) \; .$$

The initial state is $|\uparrow\rangle$. The stationary states evolve with time with the factor $\exp(\mp i\omega t)$. So at time t_1, the state is

$$|t_1\rangle = \frac{1}{\sqrt{2}}\left(|+\rangle\, e^{-i\omega t} + |-\rangle\, e^{+i\omega t}\right) ,$$

and taking the dot product with $|\downarrow\rangle$, we find the amplitude to be in state $|\downarrow\rangle$ is

$$\langle\downarrow|\, t_1\rangle = \frac{1}{2}\left(e^{-i\omega t_1} - e^{+i\omega t_1}\right) = -i\sin\omega t_1 ,$$

so the probability to be in state $|\downarrow\rangle$ at t_1 is $\sin^2\omega t_1$. Similarly, the probability to be in state $|\uparrow\rangle$ is $\cos^2\omega t_1$.

End Solution

(b) The spin, starting from whatever state the measurement at t_1 left it in (either $|\uparrow\rangle$ or $|\downarrow\rangle$), continues to evolve in the magnetic field to time $t_2 = t_1 + \tau$. At t_2, the z-component of the spin is measured again. What is the probability of finding the electron in state $|\downarrow\rangle$ for this measurement?

Solution

There are two ways the electron can get to t_2 with spin down: (1) it can have spin up at the first measurement and be found with spin down at the second measurement (probability: $\cos^2\omega t_1 \sin^2\omega\tau$), or (2) if can have spin down at both measurements (probability: $\sin^2\omega t_1\cos^2\omega\tau$). These two ways of getting to t_2 with spin down are independent, so we add the probabilities (not the amplitudes) and we find

$$P(\downarrow, t_2) = \cos^2\omega t_1 \sin^2\omega\tau + \sin^2\omega t_1 \cos^2\omega\tau .$$

End Solution

(c) Starting again from $t = 0$ in state $|\uparrow\rangle$, measurements are made repeatedly at intervals of $\tau \ll 1/\omega$ up to time $T \gg \tau$ and with $T\omega \approx 1$. That is, measurements are made at $t_n = n\tau$ with $n = 1, 2, 3, \ldots, T/\tau$. Estimate τ such that the probability of being in the state $|\downarrow\rangle$ at $t = T$ is less than $p = 0.01$. Hint: the key word here is *estimate*.

Solution

To be in the down state at T, the spin must have flipped an odd number of times, j, and not flipped $n-j$ times. The probability of flipping in an interval is $\sin^2\omega\tau \approx \omega^2\tau^2$. The probability of not flipping in an interval is $1-\sin^2\omega\tau \approx 1-\omega^2\tau^2$. There are n arrangements with one flip, $n(n-1)(n-2)/3!$ arrangements with three flips, $n(n-1)(n-2)(n-3)(n-5)/5!$ arrangements with five flips, ... The probability of being in the state $|\downarrow\rangle$ at T is

$$P \approx \sum_{j\ \mathrm{odd}} \frac{n!}{j!(n-j)!}(\omega^2\tau^2)^j(1-\omega^2\tau^2)^{n-j} \approx n\omega^2\tau^2 = \omega T\omega\tau \approx \omega\tau .$$

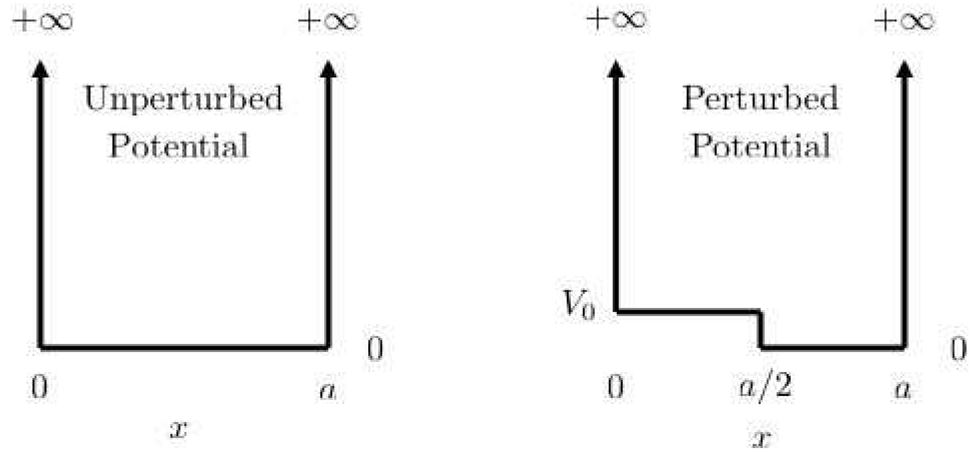
So, we must have $\tau < p/\omega = 0.01/\omega$.

This problem was adapted from a problem on the January, 2011 Prelims.

End Solution

2. Perturbed boxes. A particle of mass m moves in one dimension and is confined to a "box" by the potential $V = 0$ for $0 < x < a$ and $V = \infty$ for $x < 0$ or $x > a$.

(a) Prior to $t = 0$, the particle is in its ground state with energy E_1. From $t = 0$ to $t = T$, a perturbation is applied: for $0 < x < a/2$, the potential is changed from 0 to V_0 with $V_0 \ll E_2 - E_1$, where E_2 is the energy of the first excited state of the particle in the unperturbed potential well. After $t = T$, the potential is the same as it was before



$t = 0$. Find the probability that for $t > T$ the particle is in the first excited state (of the unperturbed potential) to the lowest non-vanishing order in $V_0/(E_2 - E_1)$. Hint: rather than blindly applying the golden rule, work this out from perturbation theory and Schroedinger's equation!

Solution

To start with, we should remember the energies and stationary states of the unperturbed potential. The stationary states are standing waves with nodes at either end of the box so there are an integral number of half wavelengths across the box. This means $k_n = n\pi/a$ where $n = 1, 2, 3, \ldots$. So, $\phi_n(x) = (2/a)^{1/2}\sin(n\pi x/a)$ and $E_n = n^2\pi^2\hbar^2/2ma^2$. We write

$$\psi(t) = \sum_n c_n(t)\phi_n(x)e^{-i\omega_n t} ,$$

where the ϕ_n are the stationary states of the unperturbed potential, $\omega_n = E_n/\hbar$, and the time dependence due to the perturbation is contained in the expansion coefficients, $c_n(t)$. At $t = 0$, $c_1 = 1$ and all the rest are 0. Write the Hamiltonian as $H = H_0 + V$ where H_0 is the Hamiltonian with the unperturbed potential and V represents the perturbation. Then plug $\psi(t)$ into Schroedinger's equation:

$$i\hbar\sum_n \dot{c}_n\phi_n e^{-i\omega_n t} = V\sum_n c_n\phi_n e^{-i\omega_n t} ,$$

where the time derivative operating on the exponentials on the left hand side is canceled by H_0 operating on the right hand side. To lowest order, all expansion coefficients retain their initial values ($c_1 = 1$, all others 0). We plug this in on the right hand side and solve for $c_n(t)$ on the left hand side. We are only interested in c_2, so we take the dot product of this equation with ϕ_2 and wind up with

$$c_2(T) = \frac{1}{i\hbar}\int_0^T dt\, e^{i(\omega_2 - \omega_1)t} \langle \phi_2 \mid V \mid \phi_1 \rangle \; .$$

The matrix element is

$$\begin{aligned}\langle \phi_2 \mid V \mid \phi_1 \rangle &= V_0 \int_0^{a/2} dx\, \frac{2}{a} \sin(\pi x/a)\sin(2\pi x/a) = V_0 \frac{2}{\pi}\int_0^{\pi/2} dy\, \sin y\, \sin 2y \\ &= V_0 \frac{4}{\pi}\int_0^{\pi/2} dy\, \sin y\, \sin y\, \cos y = \frac{4V_0}{3\pi}\end{aligned} \; .$$

The rest of the transition amplitude is

$$\frac{1}{i\hbar}\int_0^T dt\, e^{i(\omega_2 - \omega_1)t} = \frac{2e^{i(\omega_2 - \omega_1)T/2}\sin(\omega_2 - \omega_1)T/2}{i(E_2 - E_1)} \; .$$

Taking the product of this with the matrix element and squaring we get

$$\text{Prob(first excited state)} = \frac{64}{9\pi^2}\frac{V_0^2}{(E_2 - E_1)^2}\sin^2((E_2 - E_1)T/2\hbar) \; .$$

End Solution

(b) Suppose instead, that a perturbation of the same shape as in part (a) is applied, but its strength is increased very slowly from zero up to an amplitude V_s much, much larger than $E_2 - E_1$. Upon reaching V_s, the perturbation is suddenly removed. In this case, what is the probability that the particle (which again was in the ground state before the perturbation was applied) winds up in the first excited state. Hint: the fact that the perturbation was applied slowly allows you to know what the state is at the instant before the perturbation is removed. Why?

Solution

With the perturbation applied slowly enough, the particle stays in the ground state of the perturbed potential. (There are no frequency components at $(E_2 - E_1)/\hbar$.) Since V_s is very large, the particle is confined to an infinite well: $a/2 < x/a$. The wave function just before the perturbation is removed is 0 for $x < a/2$ and $\sqrt{4/a}\sin(2\pi x/a)$ for $a/2 < x < a$ and since this perturbation is removed suddenly, this is the wave function just after the perturbation is removed. We project this onto the wave function of the second excited state of the unperturbed potential and square to get the probability.

$$\text{Prob(first excited state)} = \left| \int_{a/2}^{a} dx\, \sqrt{\frac{2}{a}}\sin\frac{2\pi x}{a}\sqrt{\frac{4}{a}}\sin\frac{2\pi x}{a} \right|^2 = \frac{1}{2} \; .$$

This problem was adapted from a problem on the January, 2011 Prelims.

End Solution

3. Coupled angular momentum states. A two particle system is in a state $|\psi_0\rangle$, where each particle has orbital angular momentum quantum numbers $l = 1$ and $m_l = 0$. The two particles are **not** identical. The total angular momentum is denoted by $\boldsymbol{J} = \boldsymbol{L}_1 + \boldsymbol{L}_2$ and the eigenvalues of J^2 are $\hbar^2 j(j+1)$.

(a) The two particle state may be expanded in eigenstates of J^2. What values of j have non-zero amplitude in the expansion? For each of these values, what is the probability that it will be found in a measurement of J^2?

Solution

The states of the system may be represented as the direct product of eigenstates of $\boldsymbol{L}_1$ and eigenstates of $\boldsymbol{L}_2$. That is L_1^2 and L_{1z} eigenstates and L_2^2 and L_{2z} eigenstates. Equivalently, they can be represented by eigenstates of J^2, J_z, L_1^2 and L_2^2. The are three possible states for $l = 1$. Since $l_1 = l_2 = 1$, there are nine possible states altogether. Two angular momenta with $l = 1$ can be added to give total angular momenta $j = 2$, five states, $j = 1$, three states, and $j = 0$, one state. Our job is to express total angular momentum states: $|j, m_j, l_1, l_2\rangle$ in terms of products of individual angular momentum states $|l_1, m_1\rangle\,|l_2, m_2\rangle$. Our technique will be to find the highest m_j state in a j multiplet, then successively use the ladder operator to lower m_j until we have all the states in the multiplet. Recall

$$L_\pm\,|l, m\rangle = \sqrt{l(l+1) - m(m\pm 1)}\;|l, m\pm 1\rangle \;.$$

Now, the $|2, 2, 1, 1\rangle$ state requires $m_1 = m_2 = 1$ in order that $m_j = 2$. There is only one way to get this state:

$$|2, 2, 1, 1\rangle = |1, 1\rangle\,|1, 1\rangle \;,$$

where first ket on the right hand side refers to particle 1 and the second to particle two. Operating with $J_- = L_{1-} + L_{2-}$, we have

$$\begin{aligned} J_-\,|2, 2, 1, 1\rangle &= 2\,|2, 1, 1, 1\rangle \\ &= L_{2-}\,|1, 1\rangle\,|1, 1\rangle + L_{1-}\,|1, 1\rangle\,|1, 1\rangle \;, \\ &= \sqrt{2}\,|1, 1\rangle\,|1, 0\rangle + \sqrt{2}\,|1, 0\rangle\,|1, 1\rangle \end{aligned}$$

or

$$|2, 1, 1, 1\rangle = \frac{1}{\sqrt{2}}(|1, 1\rangle\,|1, 0\rangle + |1, 0\rangle\,|1, 1\rangle) \,.$$

Applying J_- three more times, we find

$$|2, 0, 1, 1\rangle = \frac{1}{\sqrt{6}}\,(|1, 1\rangle\,|1, -1\rangle + 2\,|1, 0\rangle\,|1, 0\rangle + |1, -1\rangle\,|1, 1\rangle) \,,$$

$$|2, -1, 1, 1\rangle = \frac{1}{\sqrt{2}}(|1, 0\rangle\,|1, -1\rangle + |1, -1\rangle\,|1, 0\rangle) \,,$$

$$|2, -2, 1, 1\rangle = |1, -1\rangle\,|1, -1\rangle \;.$$

The state $|1,1,1,1\rangle$ must be a superposition of the same states that make up $|2,1,1,1\rangle$ in order to get the correct J_z. In addition $|1,1,1,1\rangle$ and $|2,1,1,1\rangle$ must be orthogonal. This determines $|1,1,1,1\rangle$ up to a phase:

$$|1,1,1,1\rangle = \frac{1}{\sqrt{2}}(|1,1\rangle\,|1,0\rangle - |1,0\rangle\,|1,1\rangle)\,.$$

Applying J_- twice, we find

$$|1,0,1,1\rangle = \frac{1}{\sqrt{2}}\,(|1,1\rangle\,|1,-1\rangle - |1,-1\rangle\,|1,1\rangle)\,,$$

$$|1,-1,1,1\rangle = \frac{1}{\sqrt{2}}(|1,0\rangle\,|1,-1\rangle - |1,-1\rangle\,|1,0\rangle)\,,$$

Finally, $|0,0,1,1\rangle$ must be a superposition of the same states that make up $|1,0,1,1\rangle$ and $|2,0,1,1\rangle$ and it must be orthogonal to both. This determines (up to a phase)

$$|0,0,1,1\rangle = \frac{1}{\sqrt{3}}\,(|1,1\rangle\,|1,-1\rangle - |1,0\rangle\,|1,0\rangle + |1,-1\rangle\,|1,1\rangle)\,.$$

Now we know the expansions of $|j,m_j,1,1\rangle$ states in terms of $|1,m_1\rangle\,|1,m_2\rangle$ states, but what we really want is the expansion of $|1,0\rangle\,|1,0\rangle$ in terms of $|j,0,1,1\rangle$ states. There are three such states:

$$|2,0,1,1\rangle = \frac{1}{\sqrt{6}}\,(|1,1\rangle\,|1,-1\rangle + 2\,|1,0\rangle\,|1,0\rangle + |1,-1\rangle\,|1,1\rangle)\,,$$

$$|1,0,1,1\rangle = \frac{1}{\sqrt{2}}\,(|1,1\rangle\,|1,-1\rangle - |1,-1\rangle\,|1,1\rangle)\,,$$

$$|0,0,1,1\rangle = \frac{1}{\sqrt{3}}\,(|1,1\rangle\,|1,-1\rangle - |1,0\rangle\,|1,0\rangle + |1,-1\rangle\,|1,1\rangle)\,.$$

Solving, we find

$$|1,0\rangle\,|1,0\rangle = \frac{1}{\sqrt{3}}(\sqrt{2}\,|2,0,1,1\rangle - |0,0,1,1\rangle)\,.$$

We might have guessed that $|2,0,1,1\rangle$ and $|0,0,1,1\rangle$ would be the only states present because these have even parity as does $|0,1\rangle\,|0,1\rangle$ which is the product of two odd parity states, while $|1,0,1,1\rangle$ has odd parity. However, to know the probabilities, we have to work out the expansion. When measuring J^2, the probability of getting $j=2$ is $2/3$, the probability of getting $j=0$ is $1/3$ and the probability of getting $j=1$ is 0.

——— End Solution ———

At $t = 0$, a coupling between the angular momenta is turned on so the system has Hamiltonian,

$$H = \gamma \boldsymbol{L}_1 \cdot \boldsymbol{L}_2 \ ,$$

where γ is a constant. The state of the system now depends on time, $|\psi(t)\rangle$.

(b) Show that $|\langle \psi(t) \mid \psi_0 \rangle|^2$ is periodic with period T and find T. Also evaluate $|\langle \psi(t) \mid \psi_0 \rangle|^2$ at $t = T/2$

Solution

$$H = \gamma \boldsymbol{L}_1 \cdot \boldsymbol{L}_2 = \frac{\gamma}{2}(J^2 - L_1^2 - L_2^2) \ ,$$

so the states $|j, m_j, l_1, l_2\rangle$ are eigenstates of the new Hamiltonian, whereas the states $|l_1, m_1\rangle\,|l_2, m_2\rangle$ are not. The expansion of the initial state in the states $|j, m_j, l_1, l_2\rangle$ was worked out in part (a). The time evolution is found by multiplying each stationary state by $\exp -iEt/\hbar$ where E is the energy of the stationary state. For $j = 2$, the expression above gives $E_2 = \gamma\hbar^2$ and for $j = 0$, $E_0 = -2\gamma\hbar^2$. So,

$$|\psi(t)\rangle = \sqrt{\frac{2}{3}}\ |2, 0, 1, 1\rangle\ e^{-i\gamma\hbar t} - \sqrt{\frac{1}{3}}\ |0, 0, 1, 1\rangle\ e^{+2\gamma\hbar t} \ ,$$

and

$$\langle \psi(t) \mid \psi_0 \rangle = \frac{2}{3}\, e^{+i\gamma\hbar t} + \frac{1}{3}\, e^{-2i\gamma\hbar t} = e^{i\gamma\hbar t}\left(\frac{2}{3} + \frac{1}{3}\, e^{-3i\gamma\hbar t}\right) \ .$$

Then

$$|\langle \psi(t) \mid \psi_0 \rangle|^2 = \frac{5}{9} + \frac{4}{9}\ \cos(3\gamma\hbar t) \ ,$$

so indeed, the probability of finding the system in the same state as $|\psi_0\rangle$ is periodic and has period

$$T = \frac{2\pi}{3\gamma\hbar} \ .$$

For $t = T/2$,

$$|\langle \psi(t) \mid \psi_0 \rangle|^2 = \frac{5}{9} + \frac{4}{9}\ \cos(\pi) = \frac{1}{9} \ ,$$

which means the state is mostly $|1, 1\rangle\,|1 - 1\rangle$ and $|1, -1\rangle\,|1, 1\rangle$.

End Solution

4. Coulomb excitation. Consider hydrogen in its ground state at $t = -\infty$. It's acted on by an electric field in the z-direction of the form

$$\boldsymbol{E}(t) = \frac{E_0 \boldsymbol{e}_z}{1 + t^2/\tau^2} \, .$$

This field can be represented by the potential $\phi = -E(t)z$. This is an approximation to what happens when a charge particle passes nearby. If it's not relativistic, we can ignore its magnetic field. What is the probability that the electron winds up in the $2p$ state at $t = +\infty$?

Useful data: for hydrogen,

$$R_{10}(r) = 2\left(\frac{1}{a}\right)^{3/2} e^{-r/a} \, ,$$

$$R_{20}(r) = 2\left(\frac{1}{2a}\right)^{3/2} \left(1 - \frac{r}{2a}\right) e^{-r/2a} \, ,$$

$$R_{21}(r) = \frac{1}{\sqrt{3}}\left(\frac{1}{2a}\right)^{3/2} \frac{r}{a} e^{-r/2a} \, ,$$

$$a = \frac{e^2}{\hbar c} \, .$$

Also,

$$\int_{-\infty}^{+\infty} \frac{e^{-i\alpha z}}{1 + z^2}\, dz = \pi e^{-|\alpha|} \, .$$

Solution

The amplitude to go from state $|m\rangle$ to state $|n\rangle$ is

$$\langle n, +\infty \mid m, -\infty \rangle = \frac{1}{i\hbar} \int_{-\infty}^{+\infty} dt \, e^{i\omega_{nm} t} \langle n \mid V(t) \mid m \rangle \, ,$$

where $\omega_{nm} = (E_n - E_m)/\hbar = (-e^2/8a + e^2/2a)/\hbar = 3e^2/(8a\hbar)$ and where the perturbing potential is $V(t) = +eE(t)z$. (The potential can be thought of as acting on the electron.) Now, $z = r\cos\theta$. The $2p$ states involve $Y_{1,+1}$, $Y_{1,0}$, and $Y_{1,-1}$. For the $m = \pm 1$ states, the integral over ϕ will give 0. So, we need to evaluate

$$\int \sin\theta \, d\theta \, d\phi \, Y^*_{1,0}(\theta, \phi) \, \cos\theta \, Y_{0,0}(\theta, \phi) = \frac{1}{\sqrt{3}} \, .$$

If you forgot the normalization of $Y_{1,0}$, it's easy to work out. The radial part requires the evaluation of

$$\int_0^\infty r^2 \, dr \, R_{21}(r) \, r \, R_{1,0}(r) = \frac{2^{15/2} a}{3^{9/2}} \, .$$

Next we need to evaluate the time integral which is really the Fourier transform of the potential at the frequency corresponding to the energy difference.

$$\int_{-\infty}^{+\infty} dt\, \frac{e^{i\omega_{nm}t} eE_0}{1+t^2/\tau^2} = \pi e E_0 \tau e^{-|\omega_{nm}|\tau} .$$

We multiply the three results together to get the transition amplitude,

$$\langle n, +\infty \mid m, -\infty\rangle = \frac{2^{15/2}\pi}{3^5 i} \frac{eE_0 a\tau e^{-|\omega_{nm}|\tau}}{\hbar} ,$$

and the transition probability is

$$P_{12} = \frac{2^{15}\pi^2}{3^{10}} \frac{(eE_0 a\tau)^2 e^{-3e^2\tau/(4a\hbar)}}{\hbar^2} .$$

This problem is partially based on a problem from Schwabl.

End Solution

5. Scattering from a spherical shell. A particle of mass m and energy $E = \hbar^2 k^2/2m$ is scattered by the fixed, spherically symmetric potential

$$V(r) = -V_0 a \delta(r-a) ,$$

where V_0 and a are a positive constants. In the following, use suitable approximations.

(a) What is the total scattering cross section at very low energies ($ka \ll 1$)?

Solution

At low energies, the scattering is spherically symmetric and only the s-wave scattering is important. In this case all we need to do is calculate the s-wave phase shift, δ_0. The total cross section is

$$\sigma = \frac{4\pi}{k^2} \sin^2 \delta_0 .$$

So the problem is to calculate δ_0. Writing the Schroedinger equation in spherical coordinates, the angular parts vanish for $l = 0$. If we let $u(r) = rR(r)$, where $R(r)$ is the radial wave function, the Schroedinger equation becomes

$$\frac{d^2u}{dr^2} - \frac{2m}{\hbar^2} V u + k^2 u = 0 .$$

The solution must vanish at $r = 0$, so for $r < a$ the solution is

$$u_i(r) = \sin kr .$$

For $r > a$, the solution must be of the form of an incoming and an outgoing wave with a phase shift for the outgoing wave. Also, the if there were no scattering, the incoming and outgoing waves must add up to a function proportional to $\sin kr$. So, the solution for $r > a$ looks like

$$\begin{aligned} u(r) &= -e^{-ikr} + e^{ikr + 2i\delta_0} \\ &= \left(-e^{-ikr - i\delta_0} + e^{ikr + i\delta_0}\right) e^{i\delta_0} \\ &= \sin(kr + \delta_0)\left(2ie^{i\delta_0}\right) . \end{aligned}$$

We can absorb the last factor in the overall normalization constant and the solution for $r > a$ can be written

$$u_o(r) = A \sin(kr + \delta_0) ,$$

where A is the normalization constant and δ_0 is the s-wave phase shift we want to find. The wave function must be continuous at $r = a$ which means

$$\sin(ka) = A \sin(ka + \delta_0) ,$$

or

$$A = \frac{\sin(ka)}{\sin(ka + \delta_0)} .$$

Since we have a δ-function potential, there is a discontinuity in the slope of the wave function at $r = a$. We find this discontinuity by integrating the Schroedinger equation from $a - \epsilon$ to $a + \epsilon$ and taking the limit as $\epsilon \to 0$. The first term gives

$$\begin{aligned} u_o' - u_i' &= Ak\cos(ka + \delta_0) - k\cos(ka) \\ &= \frac{k}{\sin(ka + \delta_0)}\left(\sin(ka)\cos(ka + \delta_0) - \sin(ka + \delta_0)\cos(ka)\right) \\ &= -\frac{k\sin\delta_0}{\sin(ka + \delta_0)} . \end{aligned}$$

The second term gives

$$-\int_{a-\epsilon}^{a+\epsilon} \frac{2m}{\hbar^2} V(r)u(r)dr = +\frac{2mV_0a}{\hbar^2}\sin(ka) .$$

The third term in the Schroedinger equaton gives zero. Putting it all together, we have

$$\frac{k\sin\delta_0}{\sin(ka + \delta_0)} = \frac{2mV_0a}{\hbar^2}\sin(ka) .$$

This is a transcendental equation for δ_0, so in principle, we've solved the problem. However, we can simplify it a bit more. Since we are assuming $ka \ll 1$, on the right hand side we can replace $\sin(ka)$ by ka. We let $2mV_0a^2\hbar^2 = (V_0/E)(ka)^2 = C$. Then

$$\sin\delta_0 = C\sin(ka + \delta_0) = C(ka)\cos\delta_0 + C\sin\delta_0 .$$

Or

$$\sin\delta_0(1-C) = C(ka)\cos\delta_0\ ,$$
$$\sin^2\delta_0\,(1-2C+C^2) = C^2(ka)^2 - C^2(ka)^2\sin^2\delta_0\ ,$$
$$\sin^2\delta_0 = \frac{C^2(ka)^2}{1-2C+C^2(1+(ka)^2)}\ .$$

Finally,

$$\sigma = \frac{4\pi}{k^2}\sin^2\delta_0 = \frac{4\pi C^2 a^2}{1-2C+C^2(1+(ka)^2)}\ ,$$

with $C = (V_0/E)(ka)^2$. Note that if C is very small, the cross section goes to $4\pi C^2 a^2$ and if C is very large, the cross section becomes $4\pi a^2$. The extrema of the expression occur when $C = 0$, where the cross section is 0 and $C = 1$ where the cross section is $4\pi/k^2$.

End Solution

(b) What is the differential cross section at very high energies ($ka \gg 1$)?

Solution

For high energies, the Born approximation is suitable. The scattering amplitude is

$$f(\theta,\phi) = -\frac{m}{2\pi\hbar^2}\int V(\boldsymbol{r}')e^{-i\Delta\boldsymbol{k}\cdot\boldsymbol{r}'}\,d^3r'\ .$$

Since the potential is spherically symmetric, there will be no ϕ dependence and we can take $\phi = 0$, $k_i = (0,0,k)$, $k_o = (k\sin\theta, 0, k\cos\theta)$. Then the magnitude of $\Delta\boldsymbol{k}$ is $2k\sin\theta/2$ and it points in the xz-plane. In order to do the integral, take the z'-direction to be along $\Delta\boldsymbol{k}$. The ϕ' integral produces 2π and we are left with

$$\begin{aligned}
f(\theta) &= -\frac{2\pi m}{2\pi\hbar^2}\int_0^\infty r'^2\,dr'\int_0^\pi \sin\theta'\,d\theta'\,(-V_0 a)\delta(r'-a)e^{-2ik\sin(\theta/2)r'\cos\theta'}\\
&= \frac{2\pi m V_0 a^3}{2\pi\hbar^2}\int_0^\pi \sin\theta'\,d\theta'\,e^{-2ik\sin(\theta/2)a\cos\theta'}\\
&= \frac{2\pi m V_0 a^2}{4\pi i\hbar^2 k\sin(\theta/2)}\,e^{-2ik\sin(\theta/2)a\cos\theta'}\Big|_0^\pi\\
&= \frac{2mV_0a^3}{\hbar^2}\frac{\sin(2ka\sin\theta/2)}{2ka\sin\theta/2}\ .
\end{aligned}$$

The differential scattering cross section is thus

$$\frac{d\sigma}{d\Omega} = \left(\frac{2mV_0a^3}{\hbar^2}\right)^2\left(\frac{\sin(2ka\sin\theta/2)}{2ka\sin\theta/2}\right)^2\ .$$

To get the total cross section, we must integrate the differential cross section over solid angle.

End Solution