

PHYSICS DEPARTMENT, PRINCETON UNIVERSITY

PHYSICS 505 FINAL EXAMINATION

January 18, 2013, 1:30–4:30pm, A06 Jadwin Hall

SOLUTIONS

This exam contains five problems. Work any three of the five problems. All problems count equally although some are harder than others. Do all the work you want graded in the separate exam books. *Indicate clearly which three problems you have worked and want graded.* I will only grade three problems. If you hand in more than three problems without indicating which three are to be graded, I will grade the first three, only!

The exam is closed everything: no books, no notes, no calculators, no computers, no cell phones, no ipods, etc.

Write legibly. If I can't read it, it doesn't count!

Put your name on all exam books that you hand in. (Only one should be necessary!!!) On the first exam book, rewrite and sign the honor pledge: *I pledge my honor that I have not violated the Honor Code during this examination.*

If you finish early, do not leave your exam books in the room. Instead, take them to Ed Groth in room 357 Jadwin Hall.

1. Crossed \mathbf{E} and \mathbf{B} fields. Consider a particle with mass m and charge e moving in uniform $\mathbf{B} = B\mathbf{e}_z$ and $\mathbf{E} = E\mathbf{e}_x$ fields with $E < B$. Use the gauge $\mathbf{A} = Bx\mathbf{e}_y$. The Hamiltonian is

$$H = \frac{1}{2m} \left(\mathbf{p} - \frac{e}{c} \mathbf{A} \right)^2 - eEx.$$

Find the eigenfunctions and eigenvalues for this Hamiltonian. Note that if you should find that the eigenfunctions involve standard functions which we already know about, you don't have to write out each one explicitly. For example, if you should find (you won't) that the eigenfunctions involve the Legendre polynomials $P_l(y)$, you can just leave $P_l(y)$ in your result rather than explicitly write out what $P_l(y)$ is. Of course, you have to say that P_l stands for a Legendre polynomial of order l .

Hint: you might start by reminding yourself what the classical motion looks like.

Solution

To start with the classical motion, there is no force in the z direction, so the z -component of velocity is constant. Note that if the particle has $v_y = -cE/B$, then the magnetic force exactly cancels the electric force and the particle moves at constant velocity in the negative y direction. If the particle has v_y different from above or a non-vanishing v_x , then the motion is circular motion in the xy -plane (clockwise as viewed from the positive z -direction) with a velocity in the $-y$ -direction at $v_y = -cE/B$. (Often called $\mathbf{E} \times \mathbf{B}$ drift.) So, we are looking for eigenfunctions which have a constant v_z (or p_z) along with circular and linear motion in the xy -plane.

Writing out the Hamiltonian, we have

$$H = \frac{p_x^2}{2m} + \frac{1}{2m} \left(p_y - \frac{e}{c} Bx \right)^2 + \frac{1}{2m} p_z^2 - eEx.$$

We see that p_y and p_z commute with the Hamiltonian, so our eigenfunctions can be eigenfunctions of p_y , p_z , and H , simultaneously. Let $p_z\psi = \hbar k_z\psi$ and $p_y\psi = \hbar k_y\psi$. Then the eigenvalue equation becomes

$$\left(\frac{1}{2m} p_x^2 + \frac{1}{2m} \left(\hbar k_y - \frac{e}{c} Bx \right)^2 - eEx \right) \psi = \left(\mathcal{E} - \frac{\hbar^2 k_z^2}{2m} \right) \psi = \mathcal{E}' \psi,$$

where we have used \mathcal{E} to stand for the energy eigenvalue to distinguish it from E and \mathcal{E}' for the energy eigenvalue with the energy due to the z motion removed.

Define $\omega = eB/mc$ and $v_0 = cE/B$. Then the time independent Schroedinger equation becomes

$$\left(\frac{1}{2m} p_x^2 + \frac{1}{2m} (\hbar k_y - m\omega x)^2 - m\omega v_0 x \right) \psi = \mathcal{E}' \psi,$$

Let's work on the terms above involving x .

$$\frac{1}{2m} (\hbar k_y - m\omega x)^2 - m\omega v_0 x = \frac{m\omega^2}{2} \left(x^2 - 2 \left(\frac{\hbar k_y}{m\omega} + \frac{v_0}{\omega} \right) x + \frac{\hbar^2 k_y^2}{m^2 \omega^2} \right).$$

Define

$$x_0 = \frac{\hbar k_y}{m\omega} + \frac{v_0}{\omega} .$$

Then

$$\begin{aligned} \frac{1}{2m} (\hbar k_y - m\omega x)^2 - m\omega v_0 x &= \frac{m\omega^2}{2} \left((x - x_0)^2 + \frac{\hbar^2 k_y^2}{m^2 \omega^2} - x_0^2 \right) \\ &= \frac{m\omega^2}{2} (x - x_0)^2 - \frac{1}{2} m v_0^2 - \hbar k_y v_0 . \end{aligned}$$

We can clean up the last two terms by adding and subtracting eEx_0 ,

$$\begin{aligned} -\frac{1}{2} m v_0^2 - \hbar k_y v_0 + eEx_0 - eEx_0 &= -\frac{1}{2} m v_0^2 - \hbar k_y v_0 + m\omega v_0 \left(\frac{\hbar k_y}{m\omega} + \frac{v_0}{\omega} \right) - eEx_0 \\ &= \frac{1}{2} m v_0^2 - eEx_0 . \end{aligned}$$

Returning to the time independent Schroedinger equation, we have

$$\left(\frac{1}{2m} p_x^2 + \frac{m\omega^2}{2} (x - x_0)^2 + \frac{1}{2} m v_0^2 - eEx_0 \right) \psi = \left(\mathcal{E} - \frac{\hbar^2 k_z^2}{2m} \right) \psi ,$$

So we see that the x motion is that of a harmonic oscillator centered at x_0 with frequency ω . The energy eigenvalues are

$$\mathcal{E}_{n,k_y,k_z} = (n + 1/2)\hbar\omega + \frac{\hbar^2 k_z^2}{2m} + \frac{1}{2} m v_0^2 - eEx_0 ,$$

with $\hbar k_y = m\omega x_0 - mv_0$ and (unnormalized) eigenfunction

$$\psi_{n,k_y,k_z} = \psi_n(x - x_0) e^{ik_y y} e^{ik_z z} ,$$

where ψ_n is the harmonic oscillator function with frequency ω and quantum number n .

The motion has an energy corresponding to the kinetic energy desired for the z motion, the kinetic energy due to the drift velocity in the y -direction and the kinetic energy due a harmonic oscillator at frequency ω in the x -direction. Since the center of oscillation is x_0 , there is also an electric potential energy $-eEx_0$. Classically, the kinetic energy due to the motion in the xy plane looks like $mv_x^2/2 + m(v_y - v_0)^2/2$ where v_x and v_y are the velocity components of the circular motion relative to the center of the circle which is moving with velocity $-v_0 \mathbf{e}_y$. Averaging over a period gives $mr^2\omega^2/2 + mv_0^2/2$ where r is the radius of the circular orbit. The cross term in the y -velocity averages to zero as does the variation in the electric potential energy as the particle oscillates in x . In the quantum case, the oscillator energy corresponds to $mr^2\omega^2/2$. Based on a problem in Schwabl.

End Solution

2. Spins in Positronium. Positronium is a bound state of an electron and its positively charged antiparticle, the positron. The antiparticle has the same spin and mass as the electron, just the opposite charge. This problem concerns the spins of the electron and positron when positronium is in its spatial ground state in a magnetic field in the z -direction. The Hamiltonian for the spins can be written as

$$H_s = \alpha \frac{4}{\hbar^2} \mathbf{S}_1 \cdot \mathbf{S}_2 + \beta \frac{2}{\hbar} (S_{1z} - S_{2z}) ,$$

where α and β are constants. The first term is analogous to the hyperfine interaction in ordinary hydrogen and the second term represents the interaction with the magnetic field. \mathbf{S}_1 is the spin operator for one of the particles, say the the electron, and \mathbf{S}_2 is the spin operator for the other particle. Each spin may be up or down along the z -axis, so there are a total of four possible states: $|\uparrow\uparrow\rangle$, $|\uparrow\downarrow\rangle$, $|\downarrow\uparrow\rangle$, and $|\downarrow\downarrow\rangle$, where the first slot represents the spin of particle 1, the electron, and the second slot represents the spin of particle 2, the positron. These are not necessarily the eigenstates of H_s .

- (a) To start with, suppose the magnetic field is zero, $\beta = 0$. What are the eigenvalues and eigenstates (as combinations of the four states listed above) of H_s for this case?

Solution

This problem is inspired by a problem on the January, 2011 prelims. Let the total spin be $\mathbf{S} = \mathbf{S}_1 + \mathbf{S}_2$. Then S^2 , S_z , S_1^2 , and S_2^2 all commute. Further, the eigenvalues of S^2 are $s(s+1)\hbar^2$ with $s = 1$ for the triplet state and $s = 0$ for the singlet state. The eigenvalues of S_z are $m\hbar$ with $m = -1, 0, 1$ for the three triplet states and $m = 0$ for the singlet state. The eigenvalues of S_1^2 and S_2^2 are both $3\hbar^2/4$. We also have

$$2\mathbf{S}_1 \cdot \mathbf{S}_2 = S^2 - S_1^2 - S_2^2 ,$$

so we can rewrite the Hamiltonian as

$$H_s = \alpha \frac{2}{\hbar^2} (S^2 - S_1^2 - S_2^2) + \beta \frac{2}{\hbar} (S_{1z} - S_{2z}) .$$

With $\beta = 0$, the eigenstates of H_s are the three degenerate triplet states: $|\uparrow\uparrow\rangle$, $(|\uparrow\downarrow\rangle + |\downarrow\uparrow\rangle)/\sqrt{2}$, and $|\downarrow\downarrow\rangle$ with eigenvalue α and the singlet state $(|\uparrow\downarrow\rangle - |\downarrow\uparrow\rangle)/\sqrt{2}$ with eigenvalue -3α .

End Solution

- (b) Now suppose the magnetic field is non-zero, $\beta \neq 0$. What are the eigenvalues of H_s for this case? The states you found in part (a) might be a good starting basis. Some of the eigenstates are messy to calculate and some are easy. So, instead of determining the eigenstates, discuss what happens to the states in the presence of a magnetic field, $\beta \neq 0$.

 Solution

We found in part (a) that the Hamiltonian (for $\beta = 0$) is diagonalized by the triplet and singlet states. However, S_{1z} and S_{2z} do not commute with S^2 . This means that when $\beta \neq 0$ we must compute the matrix elements of the Hamiltonian and transform to a diagonal basis. We have a 4×4 matrix, so it sounds like a lot of work. However, most of the elements are zero, so it's not that bad! We use the triplet and singlet states as a basis where

$$\begin{pmatrix} t_+ \\ t_- \\ t_0 \\ s_0 \end{pmatrix}$$

stands for the state

$$t_+ |1, +1\rangle + t_- |1, -1\rangle + t_0 |1, 0\rangle + s_0 |0, 0\rangle .$$

In this basis, when $\beta = 0$, the Hamiltonian is

$$H_s = \alpha \begin{pmatrix} +1 & 0 & 0 & 0 \\ 0 & +1 & 0 & 0 \\ 0 & 0 & +1 & 0 \\ 0 & 0 & 0 & -3 \end{pmatrix} .$$

To find the corrections for $\beta \neq 0$, we apply the operator $(2/\hbar)(S_{1z} - S_{2z})$ to each of the four basis states.

$$\frac{2}{\hbar} (S_{1z} - S_{2z}) |1, +1\rangle = \frac{2}{\hbar} (S_{1z} - S_{2z}) |\uparrow\uparrow\rangle = +|\uparrow\uparrow\rangle - |\uparrow\uparrow\rangle = 0 .$$

Similarly, the operator gives 0 when applied to $|1, -1\rangle$.

$$\frac{2}{\hbar} (S_{1z} - S_{2z}) |1, 0\rangle = \frac{2}{\hbar} (S_{1z} - S_{2z}) (|\uparrow\downarrow\rangle + |\downarrow\uparrow\rangle) / \sqrt{2} = 2(|\uparrow\downarrow\rangle - |\downarrow\uparrow\rangle) / \sqrt{2} = 2|0, 0\rangle .$$

$$\frac{2}{\hbar} (S_{1z} - S_{2z}) |0, 0\rangle = \frac{2}{\hbar} (S_{1z} - S_{2z}) (|\uparrow\downarrow\rangle - |\downarrow\uparrow\rangle) / \sqrt{2} = 2(|\uparrow\downarrow\rangle + |\downarrow\uparrow\rangle) / \sqrt{2} = 2|1, 0\rangle .$$

Thus, there are off diagonal terms that connect the two $m = 0$ states. The Hamiltonian is

$$H_s = \begin{pmatrix} \alpha & 0 & 0 & 0 \\ 0 & \alpha & 0 & 0 \\ 0 & 0 & \alpha & 2\beta \\ 0 & 0 & 2\beta & -3\alpha \end{pmatrix} .$$

We see that α is a double root and we have to diagonalize the lower right 2×2 submatrix. The characteristic equation is

$$\lambda^2 + 2\alpha\lambda - 3\alpha^2 - 4\beta^2 = 0 ,$$

which has solutions

$$\lambda = \pm 2\sqrt{\alpha^2 + \beta^2} - \alpha.$$

We see that when $\beta = 0$, the third and fourth eigenvalues are α and -3α as before.

The spin and magnetic moment are opposite for an electron (due to its negative charge) and parallel for a positron. Thus the states $|1, +1\rangle = |\uparrow\uparrow\rangle$ and $|1, -1\rangle = |\downarrow\downarrow\rangle$ have no net magnetic moment and their energies are unaffected by the field. The other two states contain equal amounts of $|\uparrow\downarrow\rangle$ and $|\downarrow\uparrow\rangle$. In these latter two states, the magnetic moments of the electron and positron are parallel. Since the first slot refers to the electron, $|\uparrow\downarrow\rangle$ has a magnetic moment pointing down and $|\downarrow\uparrow\rangle$ has a magnetic moment pointing up. If the magnetic field points up (which requires the correct sign for β), $|\uparrow\downarrow\rangle$ has a positive magnetic energy and $|\downarrow\uparrow\rangle$ has a negative magnetic energy. In the limit that the magnetic interaction is much stronger than the hyperfine interaction, the low energy state is $|\downarrow\uparrow\rangle$ with energy -2β and the high energy state is $|\uparrow\downarrow\rangle$ with energy $+2\beta$. When β is small, the high energy eigenstate is the triplet state (with $m = 0$) with an in phase admixture of the singlet state. The low energy eigenstate is the singlet state with an out of phase admixture of the triplet state.

End Solution

3. Electric pulse. Consider hydrogen in its ground state for $t \leq 0$. It's acted on by an electric field pulse in the x -direction of the form

$$\mathbf{E}(t) = \begin{cases} 0 & \text{if } t \leq 0 \\ E_0 e_x & \text{if } 0 < t \leq \tau \\ 0 & \text{if } \tau < t \end{cases},$$

where E_0 and τ are positive constants. Note that this field points in the x direction, not the z direction! Such a field can be produced by placing the atom between the plates of a capacitor and pulsing the capacitor. Obtain first order expressions for the probability that the atom winds up in the 2s state and the probabilities that the atom winds up in each of the three 2p states, $|n, \ell, m\rangle = |2, 1, 1\rangle, |2, 1, 0\rangle, |2, 1, -1\rangle$ for $t > \tau$.

Useful data: for hydrogen,

$$\begin{aligned} R_{10}(r) &= 2 \left(\frac{1}{a}\right)^{3/2} e^{-r/a} & Y_{00} &= +\sqrt{\frac{1}{4\pi}}, \\ R_{20}(r) &= 2 \left(\frac{1}{2a}\right)^{3/2} \left(1 - \frac{r}{2a}\right) e^{-r/2a} & Y_{10} &= +\sqrt{\frac{3}{4\pi}} \cos \theta, \\ R_{21}(r) &= \frac{1}{\sqrt{3}} \left(\frac{1}{2a}\right)^{3/2} \frac{r}{a} e^{-r/2a} & Y_{1\pm 1} &= \mp \sqrt{\frac{3}{8\pi}} \sin \theta e^{\pm i\phi}, \\ a &= \frac{\hbar^2}{me^2}. \end{aligned}$$

Solution

The amplitude to go from state $|0\rangle$ to state $|f\rangle$ is

$$\langle f, +\infty | 0, 0 \rangle = \frac{1}{i\hbar} \int_0^{+\infty} dt e^{i\omega_{f0}t} \langle f | V(t) | 0 \rangle,$$

where $\omega_{f0} = (E_f - E_0)/\hbar$ and where the perturbing potential is $V(t) = +eE(t)x$. (The potential can be thought of as acting on the electron.) Now, $x = r \sin \theta \cos \phi$. The 2s state is even in x as is the 1s ground state. So there is no amplitude to go to the 2s state. (Basically, the same selection rule that requires l to change by one unit in a radiative transition!) The 2p states involve $Y_{1,+1}$, $Y_{1,0}$, and $Y_{1,-1}$. For $m = 0$, the integral over $\cos \phi$ gives 0. For the $m = \pm 1$ states, the integral over $\cos \phi e^{\pm i\phi}$ gives π . So, we need to evaluate

$$\pi \int_0^\pi \sin \theta d\theta \left(\mp \sqrt{\frac{3}{8\pi}} \sin \theta \right) \sin \theta \sqrt{\frac{1}{4\pi}} = \mp \frac{1}{\sqrt{6}}.$$

The radial part requires the evaluation of

$$\int_0^\infty r^2 dr R_{21}(r) r R_{10}(r) = \frac{2^{15/2} a}{3^{9/2}}.$$

Next we need to evaluate the time integral which is really the Fourier transform of the potential at the frequency corresponding to the energy difference.

$$\begin{aligned} \int_0^\tau dt e^{i\omega_{f0}t} eE_0 &= \frac{eE_0}{i\omega_{f0}} \left(e^{i\omega_{f0}\tau} - 1 \right) = \frac{eE_0 e^{i\omega_{f0}\tau/2}}{i\omega_{f0}} \left(e^{i\omega_{f0}\tau/2} - e^{-i\omega_{f0}\tau/2} \right) \\ &= eE_0 e^{i\omega_{f0}\tau/2} \frac{\sin \omega_{f0}\tau/2}{\omega_{f0}/2}. \end{aligned}$$

The frequency is

$$\omega_{f0} = \frac{E_f - E_0}{\hbar} = \left(-\frac{1}{4} \frac{e^2}{2a} + \frac{e^2}{2a} \right) / \hbar = \frac{3e^2}{8\hbar a}.$$

We multiply everything together to get the transition amplitude,

$$\langle 2, 1, \pm 1, \tau | 0, 0 \rangle = \mp \frac{2^7}{3^5 i} eE_0 a e^{i\omega_{f0}\tau/2} \frac{\sin \omega_{f0}\tau/2}{\hbar \omega_{f0}/2},$$

and the transition probability is

$$P_{0 \rightarrow 2,1,\pm 1} = \frac{2^{14}}{3^{10}} \frac{(eE_0 a \sin \omega_{f0}\tau/2)^2}{(\hbar \omega_{f0}/2)^2},$$

with $\omega_{f0} = 3e^2/(8\hbar a)$. To summarize, the probability to go to $|2, 0, 0\rangle$ is 0. The probability to go to $|2, 1, 0\rangle$ is 0. The probability to go to $|2, 1, 1\rangle$ is given by the above expression as is the probability to go to $|2, 1, -1\rangle$.

End Solution

4. Tritium Decay. Tritium is the “extra-heavy” isotope of hydrogen with a nucleus containing one proton and two neutrons. The nucleus is unstable and decays to ${}^3\text{He}$ by emitting an electron and an electron type anti-neutrino. (FYI, the half life of this decay is about 12.3 years and the energy release is about 18.6 keV.) Suppose a tritium atom is in its atomic ground state when the nuclear decay occurs. Assume that the decay is sufficiently rapid that the wave function of the atomic electron is unchanged by the decay and that the change of mass and the recoil of the nucleus are can be neglected. In other words, the only effect on the atomic electron is the sudden change in the charge of the nucleus, from $+e$ to $+2e$.

- (a) With an electron moving in the Coulomb potential, the ground state wave function is $\psi_0(r, \theta, \phi) = Ae^{-\kappa r}$. Determine A and κ for hydrogen and for singly ionized helium (the result of the decay).

Solution

This problem is based on a problem in the May, 2012 prelims. The Hamiltonian is

$$H = \frac{p^2}{2m} - \frac{Ze^2}{r},$$

where m is the mass of the electron or, more properly, the reduced mass of the system. The change in the reduced mass due to the decay is tiny, so we ignore it! Since we are dealing with ground states, $L = 0$ and we only need the radial momentum operator. In this case, we can write the Hamiltonian as

$$H = -\frac{\hbar^2}{2m} \left(\frac{1}{r} \frac{\partial^2}{\partial r^2} r \right) - \frac{Ze^2}{r}.$$

Of course, the eigenvalue equation (with subscripts indicating the ground state) is

$$\begin{aligned} H\psi_0 &= E_0\psi_0 \\ \left(-\frac{\hbar^2}{2m} \left(\frac{1}{r} \frac{\partial^2}{\partial r^2} r \right) - \frac{Ze^2}{r} \right) Ae^{-\kappa r} &= E_0 Ae^{-\kappa r} \\ -\frac{\hbar^2}{2m} \left(-\frac{2\kappa Ae^{-\kappa r}}{r} \right) - \frac{\hbar^2}{2m} \left(\kappa^2 Ae^{-\kappa r} \right) - \frac{Ze^2}{r} Ae^{-\kappa r} &= E_0 Ae^{-\kappa r}. \end{aligned}$$

The two terms proportional to $1/r$ on the left hand side of this equation must add up to zero which immediately leads to

$$\kappa = \frac{Zme^2}{\hbar^2} = \frac{Z}{a},$$

where $a = \hbar^2/me^2$ is the Bohr radius. Also, we find

$$E_0 = -\frac{\hbar^2}{2m} \kappa^2 = -\frac{1}{2} \frac{Z^2 e^2}{a}.$$

To get A , we require the wave function to be normalized.

$$\begin{aligned}
 1 &= \int r^2 dr d\Omega A^2 e^{-2\kappa r} \\
 &= \frac{4\pi A^2}{8\kappa^3} \int_0^\infty x^2 e^{-x} dx \\
 &= \frac{4\pi A^2}{8\kappa^3} \left(-x^2 e^{-x} - 2x e^{-x} - 2e^{-x} \right) \Big|_0^\infty \\
 &= \frac{\pi A^2}{\kappa^3},
 \end{aligned}$$

from which we infer that, up to a phase,

$$A = \frac{1}{\sqrt{\pi}} \left(\frac{Z}{a} \right)^{3/2}.$$

Note that these results are identical to the lecture derived hydrogen-like atom ground state (what did you expect?). Finally, for hydrogen,

$$\kappa = \frac{1}{a} \quad \text{and} \quad A = \frac{1}{\sqrt{\pi}} \left(\frac{1}{a} \right)^{3/2},$$

and for singly ionized helium,

$$\kappa = \frac{2}{a} \quad \text{and} \quad A = \frac{1}{\sqrt{\pi}} \left(\frac{2}{a} \right)^{3/2}.$$

End Solution

(b) What is the probability that immediately after the decay of the nucleus, the atomic electron is in the ionized helium ground state?

Solution

The state of the system immediately after the decay is not an eigenfunction of the new Hamiltonian, but it can be expanded in eigenfunctions of this Hamiltonian. The expansion coefficients are given by

$$c_n = \langle \varphi_n | \psi_0 \rangle,$$

where ψ_0 is the ground state of the hydrogen atom—the wave function immediately after the decay, while $\{\varphi_n\}$ are the eigenfunctions of singly ionized helium. The amplitude to be in the ground state is then

$$\begin{aligned}
 c_0 &= \int r^2 dr d\Omega \frac{1}{\sqrt{\pi}} \left(\frac{2}{a} \right)^{3/2} e^{-2r/a} \frac{1}{\sqrt{\pi}} \left(\frac{1}{a} \right)^{3/2} e^{-r/a} \\
 &= \frac{8\sqrt{2}}{27} \int_0^\infty x^2 dx \\
 &= \frac{16\sqrt{2}}{27}.
 \end{aligned}$$

The probability to be in the ground state is

$$P_0 = |c_0|^2 = \frac{512}{729} = 0.702.$$

End Solution

(c) What is the expectation value of the energy of the atomic electron in the ionized helium immediately after the decay?

Solution

We need to compute $\langle \psi_0 | H_{\text{He}} | \psi_0 \rangle$, where ψ_0 is the ground state of the hydrogen atom (the state right after the decay) and H_{He} is the Hamiltonian for the singly ionized helium. Since

$$H_{\text{He}} = H_{\text{H}} - \frac{e^2}{r},$$

(with H_{H} the Hamiltonian for hydrogen) we really only need to compute $\langle \psi_0 | -e^2/r | \psi_0 \rangle$ and add it to the ground state energy of the hydrogen atom. If we remember that the kinetic energy is half the absolute value of the potential energy, we know that the kinetic term in the ground state of hydrogen is $e^2/2a$ and the potential term is $-e^2/a$ giving a total of $-e^2/2a$ for the ground state energy. We just need to double the potential energy contribution and get $-3e^2/2a$ for the expectation value of the energy following the decay. Of course, if you think this is too easy, you are certainly welcome to do the integrals! We already computed (in part (a)) the ground state energy of hydrogen, so we want to compute

$$\begin{aligned} \left\langle -\frac{e^2}{r} \right\rangle &= \frac{1}{\pi a^3} \int r^2 dr d\Omega e^{-r/a} \left(-\frac{e^2}{r} \right) e^{-r/a} \\ &= -\frac{4e^2}{4a} \int_0^\infty x dx e^{-x} \\ &= -\frac{e^2}{a}. \end{aligned}$$

Adding this to the ground state energy for hydrogen we get the same answer as above

$$\langle \psi_0 | H_{\text{He}} | \psi_0 \rangle = -\frac{3e^2}{2a}.$$

End Solution

5. A non-relativistic particle of mass m scatters from a spherical square well given by the potential $V(r) = -V_0 < 0$ for $r < a$ and $V(r) = 0$ for $r > a$. We are considering low energy scattering and you may take the limit $\lambda/a \rightarrow \infty$ at the appropriate point in your calculation. λ is the de Broglie wavelength of the particle.

(a) In this limit obtain the differential cross section $d\sigma/d\Omega$ and the total cross section σ .

Solution

This problem is based on a problem from the May, 2012 prelims. Since we have low energy scattering from a spherically symmetric scatterer, we use the method of partial waves. Since we are going to extreme low energy (0!), we need only the s -wave. For 0 angular momentum the Schroedinger equation is

$$-\frac{\hbar^2}{2m} \frac{1}{r} \frac{d^2}{dr^2} (rR(r)) + V(r)R(r) = ER(r),$$

where $R(r)$ is the radial wave function which only needs $1/\sqrt{4\pi}$ for the angular part to be the complete wave function. We convert this into a standard 1D equation by the trick of substituting $u(r) = rR(r)$ to get

$$-\frac{\hbar^2}{2m} \frac{d^2 u}{dr^2} + Vu = Eu.$$

For $r < a$, the equation is

$$\frac{d^2 u}{dr^2} + \frac{2m}{\hbar^2} (V_0 + E)u = 0,$$

or, with $q = \sqrt{2m(V_0 + E)/\hbar^2}$,

$$\frac{d^2 u}{dr^2} + q^2 u = 0,$$

whose solution is $u = \sin(qr)$. The cosine solution would lead to a singular $R(r)$ at $r = 0$ and is therefore not allowed. Let $k = \sqrt{2m(E)/\hbar^2}$. Then the exterior solution is

$$u = -e^{-ikr} + e^{ikr} + 2i\delta_0 = 2ie^{i\delta_0} \sin(kr + \delta_0),$$

where δ_0 is the s -wave phase shift and the sign on the first term ensures we have the non-singular solution in the limit of no scattering ($V_0 \rightarrow 0$, $\delta_0 \rightarrow 0$). The phase shift is found by matching logarithmic derivatives at $r = a$:

$$\left. \frac{d}{dr} \log(\sin(qr)) \right|_{r=a} = \left. \frac{d}{dr} \log \left(2ie^{i\delta_0} \sin(kr + \delta_0) \right) \right|_{r=a},$$

or

$$\frac{\tan(qa)}{qa} = \frac{\tan(ka + \delta_0)}{ka}.$$

Which leads to

$$\delta_0 = \tan^{-1} \left(\frac{ka \tan(qa)}{qa} \right) - ka .$$

Since we will be taking the limit $ka \rightarrow 0$, we can write

$$\delta_0 = ka \left(\frac{\tan(qa)}{qa} - 1 \right) .$$

The total cross section is

$$\sigma = \sigma_0 = \frac{\pi(2\ell + 1)}{k^2} 4 \sin^2 \delta_0 = 4\pi a^2 \left(\frac{\tan(q_0 a)}{q_0 a} - 1 \right)^2 ,$$

where the right hand expression is for $ka \rightarrow 0$ and $q_0 = \sqrt{2mV_0/\hbar^2}$ is the interior wave number for $ka \rightarrow 0$. Also, since the scattering is spherically symmetric,

$$\frac{d\sigma}{d\Omega} = \frac{\sigma}{4\pi} .$$

End Solution

- (b) This limiting zero-energy cross section diverges to ∞ for certain values of V_0 . What are these values of V_0 and what is the physical significance of such divergences?

Solution

By inspection the cross section diverges whenever $\tan(qa) = \pm\infty$ or $qa = (2n-1)(\pi/2)$ where $n \geq 1$ is an integer. The physical significance is that there is a bound state with zero binding energy at this value for the potential. The interior wave function u reaches a maximum or minimum at $r = a$ and attaches to a constant wave function for $r > a$. If V_0 is increased by a tiny amount, the interior wave function will go just beyond the maximum/minimum and attach to an exponentially decaying wave with (since 0 energy) an infinite decay length (or if not quite zero energy, a very long decay length!). One can imagine starting with V_0 close to zero (no bound states) and cranking it up. Each time the low energy cross section goes through a maximum, the potential has become deep enough to add another bound state, where the number of bound states corresponds to n above.

End Solution
