PHYSICS DEPARTMENT, PRINCETON UNIVERSITY

PHYSICS 505 MIDTERM EXAMINATION

October 22, 2009, 11:00am–12:20pm, Jadwin A06

SOLUTIONS

This exam contains two problems. Work both problems. They count equally although they may not be the same difficulty.

Do all the work you want graded in the separate exam books.

The exam is closed everything: no books, no notes, no calculators, no computers, no cell phones, no ipods, etc.

Write legibly. If I can't read it, it doesn't count!

Put your name on all exam books that you hand in. (Only one should be necessary!!!) On the first exam book, rewrite and sign the honor pledge: *I pledge my honor that I have not violated the Honor Code during this examination.*

1. Well! Consider an infinitely deep (one-dimensional) potential well, $V(x) = 0$ for $-a \le x \le +a$ and $V(x) = \infty$ for $x > |a|$. A particle of mass m moves in this potential well.

(a) The energy eigenvalues and time independent eigenfunctions can be labeled by an integer $n, n = 1, 2, 3, \ldots$, where the energy increases as n increases. What are the eigenvalues, E_n, and the corresponding (normalized) eigenfunctions, $\psi_n(x)$? What is the interpretation of n?

——— Solution ———

The time independent Schroedinger equation within the well is just

$$-\frac{\hbar^2}{2m}\frac{d^2}{dx^2}\psi_n = E_n\psi_n \; .$$

The wave function must be zero at the boundaries and beyond since the potential is infinite there. Solutions may be divided into even and odd parity solutions. The even solutions are $\psi_n(x) = \cos(k_n x)$ and the odd solutions are $\psi_n = \sin(k_n x)$ where $E_n = \hbar^2 k_n^2/2m$ and k_n must be chosen to satisfy the boundary conditions. For the even solutions $\cos(k_n a) = 0$ which means $k_n = n\pi/2a$ where n is an odd integer. For the odd solutions, $\sin(k_n a) = 0$, so $k_n = n\pi/2a$ where n is an even integer. The normalization integral is

$$A^2 \int_{-a}^{+a} dx \left\{ \begin{matrix} \cos^2(k_n x) \\ \sin^2(k_n x) \end{matrix} \right\} = A^2 \frac{2a}{2} \; ,$$

so $A = 1/\sqrt{a}$ times an arbitrary phase factor which we take to be 1.

To summarize, n is the number of half wavelengths across the well and

$$k_n = \frac{n\pi}{2a} \; , \quad E_n = \frac{\hbar^2 k_n^2}{2m} = \frac{\hbar^2\pi^2 n^2}{8ma^2} \; , \quad \psi_n(x) = \frac{1}{\sqrt{a}} \left\{ \begin{matrix} \cos(k_n x) \; , n = 1, 3, 5, \ldots \\ \sin(k_n x) \; , n = 2, 4, 6, \ldots \end{matrix} \right. \; ,$$

and of course, the wave functions are zero outside the well.

——— End Solution ———

(b) At $t = 0$, the system is prepared in the superposition state $\psi(x, 0) = (\psi_1(x) + \psi_2(x))/\sqrt{2}$, where ψ_1 and ψ_2 are the wave functions of the ground state and first excited state you found in part (a). What is $\psi(x, t)$ and what is the time dependence of the expectation value of x, $\langle x(t) \rangle$, when the system is in state $\psi(x, t)$? (The trig identity

$$\cos A \sin B = (\sin(A + B) - \sin(A - B))/2 \; ,$$

will be useful.)

——— Solution ———

The time dependence of each eigenstate is just $e^{-i\omega_n t}$ where $\omega_n = E_n/\hbar$, so

$$\psi(x, t) = \frac{1}{\sqrt{2a}} \left(\cos(k_1 x) e^{-i\omega_1 t} + \sin(k_2 x) e^{-i\omega_2 t} \right) \; .$$

To calculate $\langle x \rangle$, we have

$$
\begin{aligned}
\langle x(t) \rangle &= \int_{-a}^{+a} dx\, \psi^*(x,t)\, x\, \psi(x,t) \\
&= \int_{-a}^{+a} dx\, \frac{x}{2a} \left(\cos^2(k_1 x) + \sin^2(k_2 x) + \cos(k_1 x) \sin(k_2 x) \left(e^{+i\Delta\omega t} + e^{-i\Delta\omega t} \right) \right) \\
&\qquad \text{where } \Delta\omega = \omega_2 - \omega_1 \\
&= \int_{-a}^{+a} dx\, \frac{x}{a} \cos(k_1 x) \sin(k_2 x) \cos(\Delta\omega t) \qquad \text{dropping odd terms} \\
&= \int_{-a}^{+a} dx\, \frac{x}{2a} \left(\sin(k_1 + k_2)x - \sin(k_1 - k_2)x \right) \cos(\Delta\omega t) \qquad \text{using the trig identity} \\
&= \int_{0}^{+a} dx\, \frac{x}{a} \left(\sin(k_1 + k_2)x - \sin(k_1 - k_2)x \right) \cos(\Delta\omega t) \\
&= \int_{0}^{+a} dx\, \frac{x}{a} \left(\sin(3\pi x/2a) + \sin(\pi x/2a) \right) \cos(\Delta\omega t) \\
&= \frac{4a}{\pi^2} \cos(\Delta\omega t) \int_0^{\pi/2} dy\, y\, \left(\sin(3y) + \sin(y) \right) \qquad \text{substitute } y = \pi x/2a \\
&= \frac{4a}{\pi^2} \cos(\Delta\omega t) \left(-y \left(\cos(3y)/3 + \cos y \right) \Big|_0^{\pi/2} + \int_0^{\pi/2} dy\, \left(\cos(3y)/3 + \sin(y) \right) \right) \\
&\qquad \text{integration by parts} \\
&= \frac{4a}{\pi^2} \cos(\Delta\omega t) \left(\sin(3y)/9 + \sin y \right) \Big|_0^{\pi/2} \\
&= \frac{32a}{9\pi^2} \cos(\Delta\omega t) \\
&= \frac{32a}{9\pi^2} \cos\left(\frac{3\pi^2 \hbar t}{8ma^2} \right) .
\end{aligned}
$$

——————— End Solution ———————

(c) The square well is "enhanced" by the addition of a δ-function potential at the center of the well, so the potential is $V(x) = \lambda\delta(x)$ for $-a \le x \le +a$ and $V(x) = \infty$ for $|x| > a$, with $\lambda \ge 0$. Some of the eigenvalues and eigenfunctions you found in part (a) are unaffected by this addition to the potential. Others are changed. Which are unaffected and why? For those that are changed, find a transcendental equation whose solution would allow you to determine the new energy eigenvalues. By considering the behavior of this equation as a function of λ, show that, although the altered potential changes the energy eigenvalues, it does not change the ordering of the energy eigenvalues.

Solution

The eigenfunctions now must satisfy a matching condition at $x = 0$ as well as the boundary conditions at $x = \pm a$. In particular the eigenfunction must be continuous at $x = 0$ and it must have a discontinuity in slope given by integrating the δ-function. The eigenvalues and eigenfunctions that are odd in x, $n = 2, 4, 6, \ldots$ are unchanged. This is because they vanish at $x = 0$, so the integral over the δ-function gives 0 and the function and its derivative are continuous at the origin, just as they were in part (a), so there is no change.

For the even functions, we write $\psi_n(x) = \sin(k_n(a - |x|))$ which automatically satisfies the boundary conditions at $x = \pm a$. Note that $E_n = \hbar^2 k_n^2/2m$. It also satisfies the continuity condition at $x = 0$. The discontinuity in slope is found by integrating the Schroedinger equation from $-\epsilon$ to $+\epsilon$, or

$$\int_{-\epsilon}^{+\epsilon} dx \left(\lambda\, \delta(x)\, \psi_n(x) - E_n \psi_n(x) = \frac{\hbar^2}{2m} \frac{d^2\psi_n}{dx^2} \right) ,$$

or

$$\frac{2m\lambda}{\hbar^2}\, \psi_n(0) = \frac{d\psi_{n+}}{dx} - \frac{d\psi_{n-}}{dx} ,$$

or

$$\frac{2m\lambda}{\hbar^2} \sin(k_n a) = -2k_n \cos(k_n a) .$$

Rearranging:

$$-k_n a\, \cot(k_n a) = \frac{m\lambda a}{\hbar^2} .$$

If $\lambda = 0$, we have the case of part (a). The solutions occur where $\cot(k_n a) = 0$ or $k_n = n\pi/2a$ with n an odd integer; these are the wave numbers for the even eigenfunctions of part (a). As λ increases from 0, $k_n a$ must increase so that $\cot(k_n a)$ becomes negative and the left hand side of the above equation is positive. As $\lambda \to \infty$, $k_n a \to (n+1)\pi/2$ but never quite gets there as long as λ is finite. Note that the wave number for the next higher energy eigenfunction is $k_{n+1} = (n+1)\pi/2a$. So for a given $\lambda > 0$, the energies and wave numbers of the odd eigenfunctions are unchanged while the energies and wave numbers of the even eigenfunctions increase but never pass those of the next higher odd eigenfunction,

End Solution

2. "Top hat" momentum distribution. A free particle of mass m moves in one dimension. The distribution of wave numbers is uniform between $k_0 - q$ and $k_0 + q$ and 0 for $|k - k_0| > q$. That is, the wave function in wave number space is

$$\varphi(k) = \begin{cases} 1/\sqrt{2q}\,, & |k - k_0| \le q; \\ 0\,, & |k - k_0| > q. \end{cases}$$

(a) What is the uncertainty in the momentum?

Solution

$$(\Delta k)^2 = \frac{1}{2q}\int_{k_0-q}^{k_0+q} dk(k-k_0)^2 = \frac{q^2}{3}\ ,$$

so $\Delta p = \hbar q/\sqrt{3}$.

End Solution

(b) What is the configuration space wave function, $\psi(x)$, at $t = 0$?

Solution

$$\begin{aligned}\psi(x) &= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{+\infty} dk\,\varphi(k)\,e^{ikx} &&= \frac{1}{\sqrt{4\pi q}}\int_{k_0-q}^{k_0+q} dk\,e^{ikx}\\ &= \frac{e^{ik_0x}}{\sqrt{4\pi q}}\int_{k_0-q}^{k_0+q} dk\,e^{i(k-k_0)x} &&= \frac{e^{ik_0x}}{\sqrt{4\pi q}}\frac{1}{ix}\left(e^{+iqx}-e^{-iqx}\right)\\ &= e^{ik_0x}\sqrt{\frac{q}{\pi}}\frac{\sin(qx)}{qx}\end{aligned}$$

End Solution

(c) The uncertainty in position (at $t = 0$) is formally infinite. This arises from the unrealistic “sharp” corners in the momentum distribution which produces large wings in the configuration space probability density. For a more realistic momentum distribution, the wings would fall off faster and the mean square position would converge. Ignoring the large wings of the configuration space probability density, make a heuristic estimate of the position uncertainty. What is the momentum—position uncertainty product? Does this agree with the limit from the uncertainty principle?

Solution

The probability density is

$$|\psi(x)|^2 = \frac{q}{\pi}\left(\frac{\sin(qx)}{qx}\right)^2\ .$$

The function in parentheses is 1 at $x = 0$ and has its first 0 at $qx = \pm\pi$. Ignoring the wings beyond this, the half width at half maximum is $qx \approx \pi/2$ and we take the corresponding x to be the position uncertainty. So $\Delta x \approx \pi/2q$. Then

$$\Delta x\Delta p \approx \frac{\hbar}{2}\frac{\pi}{\sqrt{3}} > \frac{\hbar}{2}\ ,\quad \text{the uncertainty principle limit}\ .$$

So yes, the uncertainty product agrees with the uncertainty principle limit!

End Solution