1. More H atom expectation values. For the relativistic corrections with the Coulomb potential, we needed to know $\langle 1/r^2 \rangle$ and $\langle 1/r^3 \rangle$. These can be calculated from the radial wave functions, but there is an easier way. We start with radial wave functions $u(r) = rR(r)$ and rewrite the radial Schroedinger equation in the dimensionless variable $y = r/a$.

(a) Show that the resulting equation is

$$Hu(y) = \epsilon u(y) \, ,$$

with

$$\epsilon = -\frac{Z^2}{(N + l + 1)^2} \, ,$$

and

$$H = -\frac{d^2}{dy^2} + \frac{l(l+1)}{y^2} - \frac{2Z}{y} \, .$$

(b) To find $\langle 1/r^2 \rangle$ we just find $\langle 1/y^2 \rangle / a^2$ using the above $u(y)$, H, and ϵ and we evaluate $\langle 1/y^3 \rangle / a^3$ to find $\langle 1/r^3 \rangle$. Find $\langle 1/r^2 \rangle$. Hint: the trick is to differentiate the Schroedinger equation you obtained in part (a) with respect to l.

(c) Find $\langle 1/r^3 \rangle$. Hint: this time the trick is to differentiate with respect to y.

2. What the exchange term exchanges! Consider an excited state in helium in which the Coulomb shift plus exchange shift in the energy in the levels (relative to the hydrogen like levels with energy E) is

$$\Delta E = J - \frac{1}{2}(1 + \boldsymbol{\sigma}_1 \cdot \boldsymbol{\sigma}_2)K \, ,$$

where J is the Coulomb repulsion between the two electrons and K is the "exchange term." Suppose the initial spin state is $|\uparrow\rangle \, |\downarrow\rangle$. That is, electron 1 spin is up and electron 2 spin is down. What is the time evolution of the spin state? Hint: write the spin state as a superposition of the singlet and triplet spin states.