1. Probability conservation (based on a problem in Schwabl). Recall that the Hamiltonian for a charged particle (charge $e$) of mass $m$ in an electromagnetic field described by the potentials $\phi(\boldsymbol{x}, t)$ and $\boldsymbol{A}(\boldsymbol{x}, t)$ is,

$$H = \frac{1}{2m}\left(\boldsymbol{p} - \frac{e}{c}\boldsymbol{A}\right)^2 + e\phi\,.$$

Show that a wave function $\psi$ which is a solution of Schroedinger's equation with this Hamiltonian, satisfies the probability continuity condition,

$$\frac{\partial}{\partial t}(\psi^*\psi) + \nabla\cdot\boldsymbol{j} = 0\,,$$

with $\boldsymbol{j}$ defined as

$$\begin{aligned}\boldsymbol{j} &= \frac{\hbar}{2mi}\left(\psi^*(\nabla\psi) - (\nabla\psi^*)\psi - \frac{2ie}{\hbar c}\boldsymbol{A}\psi^*\psi\right)\\ &= \frac{1}{2m}\left(\psi^*\left(\frac{\hbar}{i}\nabla - \frac{e}{c}\boldsymbol{A}\right)\psi + \psi\left(\frac{\hbar}{-i}\nabla - \frac{e}{c}\boldsymbol{A}\right)\psi^*\right)\,.\end{aligned}$$

Note that $\boldsymbol{A}$ is assumed to be real.

2. Crossed $\boldsymbol{E}$ and $\boldsymbol{B}$ fields (based on a problem in Schwabl). Consider a particle with mass $m$ and charge $e$ moving in uniform $\boldsymbol{B} = B\boldsymbol{e}_z$ and $\boldsymbol{E} = E\boldsymbol{e}_x$ fields with $E < B$. Use the gauge $\boldsymbol{A} = Bx\boldsymbol{e}_y$. The Hamiltonian is

$$H = \frac{1}{2m}\left(\boldsymbol{p} - \frac{e}{c}\boldsymbol{A}\right)^2 - eEx\,.$$

Find the eigenfunctions and eigenvalues for this Hamiltonian. Note that if you should find that the eigenfunctions involve standard functions which we already know about, you don't have to write out each one explicitly. For example, if you should find (you won't) that the eigenfunctions involve the Legendre polynomials $P_l(y)$, you can just leave $P_l(y)$ in your result rather than explicitly write out what $P_l(y)$ is. Of course, you have to say that $P_l$ stands for a Legendre polynomial of order $l$.

Hint: you might start by reminding yourself what the classical motion looks like.

3. Magnetic barrier. This problem appeared on the May, 2002 prelims. The $\Theta$ functions used below are defined so that $\Theta(x) = 0$ if $x < 0$, $\Theta(x) = 1$ if $x > 0$ and $\Theta(0) = 1/2$. In other words, $\Theta$ is the unit step function and $\Theta(x) = \int_{-\infty}^{x}\delta(x')\,dx'$. Also you may find problem 1 in this assignment to be useful.

Consider a charged particle moving in the $xy$-plane subject to a magnetic field $B_z = B\Theta(x)\Theta(d-x)$. The magnetic field is constant in a strip of width $d$ and zero everywhere else. We will study the problem of scattering of plane waves from this "magnetic barrier."

(a) Write down the Schroedinger Hamiltonian for this problem. You have to choose a gauge for the vector potential—choose the gauge $A_x = A_z = 0$, and also choose $A_y = 0$ for $x < 0$.

Consider the scattering problem for an electron incident from $x < 0$ and moving perpendicular to the barrier. For an incident wave $\exp(ikx)$ there will, in general be a transmitted wave $T\exp(ik_t x)$ and a reflected wave $R\exp(-ikx)$.

(b) The transmitted wave vector $k_t$ is determined by simple kinematics in terms of $k$ and $Bd$. What is that relation?

(c) For a given barrier, you will find that, below a certain critical energy $E_0$, $k_t$ is imaginary. What does this mean? Give a classical argument that leads to the same critical energy.

(d) What is the direction of the transmitted probability flux? It is **not** along the $x$-axis!

(e) Find the reflection and transmission coefficients in the limit $d \to 0$ with $Bd$ fixed.

4. In lecture, we worked out the generating function for Legendre polynomials,

$$F(x, \mu) = \sum_{l=0}^{\infty} x^l P_l(\mu) = \frac{1}{\sqrt{1 - 2x\mu + x^2}} \, .$$

(a) Use the generating function to demonstrate the following recursion relation among the polynomials,

$$(l+1)P_{l+1}(\mu) - (2l+1)\mu P_l(\mu) + lP_{l-1}(\mu) = 0 \, .$$

Hint: what happens if you differentiate the generating function and the series with respect to $x$?

(b) Demonstrate the following recursion relation among the polynomials

$$P'_{l+1}(\mu) - 2\mu P'_l(\mu) + P'_{l-1}(\mu) = P_l(\mu) \, .$$

Hint: what else can you differentiate with respect to?

(c) Use the generating function to obtain an expression for $P_l(0)$.