

1. Angular momentum uncertainty relations. A system is in the lm eigenstate of L^2 , L_z .

- (a) Show that the expectation values of $L_{\pm} = L_x \pm iL_y$, L_x , and L_y all vanish.
- (b) Determine $\Delta L_i = \sqrt{\langle L_i^2 \rangle - \langle L_i \rangle^2}$. Verify the generalized uncertainty relation holds for all pairs of angular momentum components. Comment on $\Delta L_x \Delta L_y$ for the cases $m = 0$ and $m = l$.

2. Fun with angular momentum commutators.

- (a) Suppose the vector operators \mathbf{A} and \mathbf{B} commute with each other and \mathbf{L} . Show that

$$[\mathbf{A} \cdot \mathbf{L}, \mathbf{B} \cdot \mathbf{L}] = i\hbar(\mathbf{A} \times \mathbf{B}) \cdot \mathbf{L}.$$

- (b) Suppose \mathbf{V} is a vector operator which might be a function of \mathbf{x} and \mathbf{p} , so it doesn't necessarily commute with \mathbf{L} . Show that

$$[\mathbf{L}^2, \mathbf{V}] = 2i\hbar(\mathbf{V} \times \mathbf{L} - i\hbar\mathbf{V}).$$

You might need the relation derived in lecture for a vector operator, $[L_i, V_j] = i\hbar\epsilon_{ijk}V_k$.

3. Classically, a particle moving in a spherically symmetric potential has the Hamiltonian

$$H = \frac{p_r^2}{2m} + \frac{L^2}{2mr^2} + V(r),$$

where $p_r = \mathbf{r} \cdot \mathbf{p}/r$. For quantum mechanics, we must define

$$p_r = \frac{1}{2} \left(\frac{1}{r}(\mathbf{r} \cdot \mathbf{p}) + (\mathbf{p} \cdot \mathbf{r})\frac{1}{r} \right), \quad (1)$$

with the Hermitian operator

$$p_r = \frac{\hbar}{i} \left(\frac{\partial}{\partial r} + \frac{1}{r} \right). \quad (2)$$

Show the operator defined in equation (1) is the same as that in equation (2). Show that p_r in equation (2) is Hermitian (consider $\psi(r, \theta, \phi)$ and $\varphi(r, \theta, \phi)$) and that when used in the Hamilton, p_r of equation (2) gives the correct Schroedinger equation. Also show that the operator $(\hbar/i)(\partial/\partial r)$ is not Hermitian!

4. What if we like x instead of z ?

- (a) Find the eigenfunction, ψ , of L^2 and L_x with eigenvalues $2\hbar^2$ and \hbar , respectively.

(b) Express the ψ just found as a linear combination of eigenfunctions of L^2 and L_z .

5. Algebra all the way. We used algebraic techniques in lecture to deduce that L^2 and L_z could be simultaneously diagonalized (that is, eigenfunctions could be eigenfunctions of L^2 and L_z at the same time) and that the eigenvalues are $l(l+1)\hbar^2$ and $m\hbar$, respectively, with l and m integers or half integers and with $l \geq 0$ and $m = -l, -l+1, \dots, l-1, l$. We then abandoned the algebraic technique and solved a differential equation to find the orbital angular momentum eigenfunctions. Here we will outline the use of algebraic techniques to deduce the orbital angular momentum eigenfunctions. We start by introducing $x_{\pm} = x \pm iy$.

(a) Show that the following commutation relations hold (you may use relations already derived in lecture):

$$\begin{aligned} [L_z, x_{\pm}] &= \pm \hbar x_{\pm} \\ [L_{\pm}, x_{\pm}] &= 0 \\ [L_{\pm}, x_{\mp}] &= \pm 2\hbar z \\ [L^2, x_{\pm}] &= 2\hbar x_{\pm} L_z + 2\hbar^2 x_{\pm} - 2\hbar z L_{\pm} . \end{aligned}$$

(b) Show that

$$L_z x_+ |l, l\rangle = x_+ L_z |l, l\rangle + \hbar x_+ |l, l\rangle = \hbar(l+1) |l, l\rangle ,$$

and

$$L^2 x_+ |l, l\rangle = \hbar^2 l(l+1) x_+ |l, l\rangle + 2\hbar^2 (l+1) x_+ |l, l\rangle = \hbar^2 (l+1)(l+2) x_+ |l, l\rangle .$$

This means that x_+ is the ladder or raising operator for states in which $m = l$.

So, any state $|l, m\rangle$ can be found by applying the operator x_+ to the state $|0, 0\rangle$ l times and then applying L_- $l - m$ times.

$$|l, m\rangle = C L_-^{l-m} x_+^l |0, 0\rangle ,$$

where C is a normalization constant. We can show that $|0, 0\rangle$ is independent of angle. $\mathbf{L} |0, 0\rangle = 0$, so rotating the state with $U_{\delta\varphi}$ introduced in lecture just gives back $|0, 0\rangle$. This means, $|0, 0\rangle$ must be a constant.

(c) Determine $|l, l\rangle$ (equivalently, $Y_l(\theta, \phi)$) up to a phase using the x_{\pm} . Hint: r commutes with \mathbf{L} , L^2 , and \mathbf{x} , so it is just a constant as far as all these operators are concerned.

From here one could go on to use L_- to determine (up to a phase) all the angular momentum eigenfunctions (for integer l). The normalization constants were given in lecture. However, it's unlikely that this will lead to new insights, so this problem ends here!

Appendix. Since we were somewhat rushed with the coverage of P_{lm} s and Y_{lm} s I include a few items here. Much more can be found in any reference on mathematical functions such as Abramowitz and Stegun or any quantum text.

Associated Legendre equation. After separation of variables θ and ϕ (so the solutions for ϕ are $\exp(\pm im\phi)$) the equation for θ becomes

$$\left(\frac{1}{\sin \theta} \frac{\partial}{\partial \theta} \left(\sin \theta \frac{\partial}{\partial \theta} \right) + l(l+1) - \frac{m^2}{\sin^2 \theta} \right) f(\theta) = 0.$$

This is the associated Legendre equation and it's customary to change the variable to $\mu = \cos \theta$.

$$\left((1 - \mu^2) \frac{d^2}{d\mu^2} - 2\mu \frac{d}{d\mu} + l(l+1) - \frac{m^2}{1 - \mu^2} \right) P_{lm}(\mu) = 0,$$

where the non-singular (at $\mu = \pm 1$) solution has been written as $P_{lm}(\mu)$ which is known as an associated Legendre function. When $m = 0$, the equation is known as the Legendre equation with regular solutions $P_{l0}(\mu) = P_l(\mu)$ which are Legendre polynomials.

$$P_l(\mu) = \frac{(-1)^l}{2^l l!} \left(\frac{d}{d\mu} \right)^l (1 - \mu^2)^l.$$

$P_l(\mu)$ is an l^{th} order polynomial in μ and is either even or odd depending on whether l is even or odd. The normalization (be careful when consulting references, not everyone uses the same) is $P_l(1) = 1$. The associated Legendre functions are given by ($m \geq 0$),

$$P_{lm}(\mu) = (1 - \mu^2)^{m/2} \left(\frac{d}{d\mu} \right)^m P_l(\mu) = \frac{(-1)^l}{2^l l!} (1 - \mu^2)^{m/2} \left(\frac{d}{d\mu} \right)^{l+m} (1 - \mu^2)^l.$$

For $m = l$, the associated Legendre function is particularly simple. The farthest right factor is a polynomial in l in which the highest power is $(-\mu^2)^l$. Since the polynomial is differentiated $2l$ times, only this term survives. The $(-1)^l$ cancels the $(-1)^l$ in front. The l derivatives produce a factor $2l!$ which combined with the other factors in front produces $(2l-1) \cdot (2l-3) \cdot (2l-5) \cdots 3 \cdot 1$ which is often abbreviated $(2l-1)!!$ where $!!$ is read "double factorial." So $P_{ll}(\mu) = (2l-1)!! (1 - \mu^2)^{l/2} = (2l-1)!! \sin^l \theta$. The associated Legendre polynomials with different l , but the same $m \geq 0$, are orthogonal on the interval -1 to $+1$,

$$\int_{-1}^{+1} d\mu P_{lm}(\mu) P_{l'm}(\mu) \frac{2}{2l+1} \frac{(l+m)!}{(l-m)!} \delta_{ll'}.$$

The normalized, orthogonal, complete eigenfunctions of L^2 and L_z (for orbital angular momentum) are the spherical harmonics which are defined in terms of the associated Legendre functions for the polar angle and the azimuthal wave for the azimuthal angle. For $m \geq 0$, these are

$$Y_{lm}(\theta, \phi) = (-1)^m \sqrt{\frac{2l+1}{4\pi} \frac{(l-m)!}{(l+m)!}} P_{lm}(\cos \theta) e^{im\phi},$$

and for negative m , we use

$$Y_{l,-m}(\theta, \phi) = (-1)^m Y_{lm}^*(\theta, \phi).$$

The ortho-normality relation is

$$\int_0^\pi \sin \theta \, d\theta \int_0^{2\pi} d\phi Y_{lm}(\theta, \phi) Y_{l'm'}^*(\theta, \phi) = \delta_{ll'} \delta_{mm'}.$$

The completeness relation is

$$\sum_{l=0}^{\infty} \sum_{m=-l}^{+l} Y_{lm}(\theta, \phi) Y_{lm}^*(\theta', \phi') = \frac{1}{\sin \theta} \delta(\theta - \theta') \delta(\phi - \phi').$$

Some of the low order spherical harmonics are,

$$\begin{aligned} Y_{00} &= +\sqrt{\frac{1}{4\pi}} \\ Y_{10} &= +\sqrt{\frac{3}{4\pi}} \cos \theta \\ Y_{11} &= -\sqrt{\frac{3}{8\pi}} \sin \theta e^{i\phi} \\ Y_{20} &= +\sqrt{\frac{5}{16\pi}} (3 \cos^2 \theta - 1) \\ Y_{21} &= -\sqrt{\frac{15}{8\pi}} \sin \theta \cos \theta e^{i\phi} \\ Y_{22} &= +\sqrt{\frac{15}{32\pi}} \sin^2 \theta e^{2i\phi} \\ Y_{30} &= +\sqrt{\frac{7}{16\pi}} (5 \cos^3 \theta - 3 \cos \theta) \\ Y_{31} &= -\sqrt{\frac{21}{64\pi}} \sin \theta (5 \cos^2 \theta - 1) e^{i\phi} \\ Y_{32} &= +\sqrt{\frac{105}{32\pi}} \sin^2 \theta \cos \theta e^{2i\phi} \\ Y_{33} &= -\sqrt{\frac{35}{64\pi}} \sin^3 \theta e^{3i\phi} \\ &\dots \end{aligned}$$