

1. From Prelims, January 12, 2007. Electron with effective mass. An electron is moving in one dimension in a potential $V(x) = 0$ for $x > 0$ and $V(x) = V_0 > 0$ for $x < 0$. The region $x > 0$ is empty space, where the electron mass is the usual bare mass m_0 , but in the region $x < 0$ it has a modified “effective mass” m_1 . When the mass of a non-relativistic particle depends on its position, the Hamiltonian should be written in the operator-ordered form

$$H = \frac{1}{2} p (m(x))^{-1} p + V(x),$$

where $[x, p] = i\hbar$.

- (a) The standard continuity conditions (continuity of $\psi(x)$ and $\psi'(x) = d\psi(x)/dx$) only apply at $x = 0$ if $m_1 = m_0$. Derive the modified continuity conditions that apply at points where the mass is discontinuous.
- (b) The (unnormalized) wavefunction of an eigenstate of the Hamiltonian with an energy $E < V_0$ is given by $\psi(x) = A \sin k(x - x_0)$ for $x > 0$. Find k , x_0 and $\psi(x)$ for $x < 0$. Make a sketch of the function $\psi(x)$, indicating its essential features.

2. Non-square Potential Barrier. In Homework 2, Problem 5, you solved the case of a plane wave incident on a barrier or well of width a . In this problem, we want to consider penetration of a barrier, so we will start out by assuming a wave of energy E is incident on a square barrier of height V_0 and width a . In your previous solution for this case, you should have discovered that the transmission probability is

$$t = \frac{4k^2\kappa^2}{(k^2 - \kappa^2)^2 \sinh^2(\kappa a) + 4k^2\kappa^2 \cosh^2(\kappa a)},$$

where $k = \sqrt{2mE/\hbar^2}$ and $\kappa = \sqrt{2m(V_0 - E)/\hbar^2}$.

- (a) Show that for a high, wide barrier, this can be written as

$$t = \frac{16E(V_0 - E)}{V_0^2} e^{-2\sqrt{2m(V_0 - E)/\hbar^2} a}.$$

It's an interesting fact that the argument of the exponent above depends on the width of the potential but the factor out in front (sometimes called the *prefactor*) doesn't. One might have thought that if one had two square barriers, each of height V_0 and width $a/2$, and these occurred one after the other with no gap, that the probability for transmission through both barriers would be the product of the probabilities for transmission through one and then the next. That is, $t(a) = t(a/2) \times t(a/2)$. This is correct for the exponential, but not for the prefactor! This can be understood in a qualitative way by noting that the exponential gives the decrease in the wave function through the barrier while the prefactor

has to do with the matching (reflecting and transmitting) at the front and back ends of the barrier.

- (b) Consider a barrier in which the potential depends on x , $V(x)$ and where the barrier extends from x_1 to x_2 . That is $V(x_1) = V(x_2) = E$ and for $x_1 < x < x_2$, $V(x) > E$. This potential is envisioned as more or less smooth with no sudden steps. Give arguments to support the idea that the transmission probability through this barrier is roughly

$$t \approx e^{-2 \int_{x_1}^{x_2} \sqrt{2m(V(x) - E)/\hbar^2} dx}.$$

3. Alpha Decay. In an alpha decay of a heavy nucleus, the alpha particle (${}^4\text{He}$ nucleus) is emitted with a typical kinetic energy of a few MeV. The half lives of alpha emitters range from a fraction of a second to billions of years. See R. D. Evans, *The Atomic Nucleus*, 1955, McGraw-Hill, p.78 for a plot in which the half life scale runs from 10^{-12} to 10^{+12} years while the decay energy runs from 4 to 7.5 MeV. The attempt to explain the large dynamic range in half life led to the tunneling explanation of alpha decay. We need a model for the potential energy of interaction between the α particle and the daughter nucleus. To start with, there is the Coulomb interaction. The daughter nucleus has a positive charge $Z_1 e$ where e is the magnitude of the electron charge and Z_1 is typically 80-90. The α particle has charge $Z_2 e$, with $Z_2 = 2$. When the α particle is outside the nucleus, it feels only the Coulomb interaction and the potential is $Z_1 Z_2 e^2 / r$ where r is the distance between the daughter nucleus and the α particle. A typical daughter nucleus has a total of $A \approx 220 - 240$ nucleons and its radius in fermis is $R \approx 1.5 A^{1/3}$. (Over the interesting range of nuclei, the radius doesn't vary much!) As the α particle approaches the edge of the nucleus it feels the strong, but short range, force from the nearby nucleons which overwhelms the Coulomb force and so the potential drops suddenly. Inside the nucleus, the α particle feels a force only from the nearest nucleons which are more or less uniformly distributed around it, leading to no net force and a flat potential. So we are led to a model with two pieces: for $r < R$, $V(r) = -V_0$ and for $r > R$, $V(r) = Z_1 Z_2 e^2 / r$. The Coulomb potential at the edge of the nucleus is $V(R) = Z_1 Z_2 e^2 / R$.

- (a) Estimate this potential height in MeV. Also estimate the radius at which the Coulomb potential is the same as the decay energy. To be definite, let's pick thorium ($Z = 90$, $A = 232$) decaying to radium ($Z = 88$, $A = 228$) with a decay energy $E \approx 4\text{MeV}$.
- (b) As a model, we assume the α particle exists as a particle within the nucleus and has the energy E . It is held within the nucleus by the barrier, but there's a chance for it to tunnel through. Calculate the transmission probability using the expression from problem 2. Be sure to make use of the fact that it's a high barrier!
- (c) Using the expression you derived above, determine a numerical value for the transmission probability in the decay of thorium to radium. You should get a very small

number. I get 3.6×10^{-40} . Estimate the lifetime of thorium. You might want to make an estimate for the number of times per second the α particle hits the barrier. This times the transition probability (per collision) is the probability per second to decay (or the inverse of the lifetime).

4. Consider an infinitely deep well, $V(x) = 0$ for $-a < x < b$ and $V(x) = \infty$ for $x < -a$ or $x > b$. Within the well, there is a δ -function addition to the potential, $V = \lambda\delta(x)$. (Centered at the origin.) This may seem a little artificial, but it's possible to confine electrons in a plane where the motion perpendicular to the plane is the motion discussed in this problem. See Schwabl, chapter 3.

- (a) Find the normalized eigenfunctions and a formula for the energy eigenvalues.
- (b) Discuss the special cases $\lambda \rightarrow 0$ and $\lambda \rightarrow \infty$.
- (c) Discuss the special case $a = b$.