1. Harmonic Oscillator Matrix Elements. We have been considering the harmonic oscillator with Hamiltonian $H = p^2/2m + m\omega^2 x^2/2$. The energy eigenstates are $|\psi_n\rangle$ with energy eigenvalues $E_n = \hbar\omega(n + 1/2)$.

(a) Compute the matrices

$$\hat{x}_{nm} = \langle \psi_n \mid x \mid \psi_m \rangle \ , \quad \hat{p}_{nm} = \langle \psi_n \mid p \mid \psi_m \rangle \ , \quad \hat{E}_{nm} = \langle \psi_n \mid H \mid \psi_m \rangle \ .$$

These are the position, momentum, and energy operators in the *energy basis* or *energy representation*. The raising and lowering operators $a^\dagger$ and a will probably be useful.

(b) Using ordinary matrix multiplication, show that $\hat{E} = \hat{p}^2/2m + m\omega^2\hat{x}^2/2$ and $\hat{x}\hat{p} - \hat{p}\hat{x} = i\hbar I$, where I is the identity matrix, $I_{nm} = \delta_{nm}$.

2. Harmonic Oscillator Coherent States. We've seen that the energy eigenstates of the quantum harmonic oscillator do not "oscillate" analogous to the classical motion. In lecture we considered a superposition of two states and showed we could get an oscillatory $\langle x \rangle$. A more elaborate construction involves *coherent* states which are eigenfunctions of the annihilation operator:

$$a\varphi_\alpha = \alpha\varphi_\alpha \ ,$$

where α is a complex number.

(a) Show that $\varphi_\alpha(x, t = 0)$ can be expanded in the energy eigenstates (and normalized) as

$$\varphi_\alpha(x, 0) = e^{-|\alpha|^2/2} \sum_{n=0}^{\infty} \frac{\alpha^n}{\sqrt{n!}} \psi_n = e^{-|\alpha|^2/2} \sum_{n=0}^{\infty} \frac{(\alpha a^\dagger)^n}{n!} \psi_0 \ .$$

(b) Now that you have the expansion in energy eigenstates, the time dependence is fairly easy to determine. Show that

$$\varphi_\alpha(x, t) = \varphi_{\alpha(t)}(x, 0) e^{-i\omega t/2} \ ,$$

where

$$\alpha(t) = \alpha(0) e^{-i\omega t} \ ,$$

and the shorthand $\varphi_{\alpha(t)}$ means: use the expression for φ_α with $\alpha(t)$ in place of α. In other words, the time dependence is found simply by multiplying the entire state by the phase factor $\exp(-i\omega t/2)$ and advancing the phase of α by $\exp(-i\omega t)$.

(c) Show that the $\langle x \rangle$ and $\langle p \rangle$ oscillate according to

$$\langle x \rangle = \sqrt{2}\, x_0\, |\alpha| \, \cos(\omega t - \delta) \ ,$$

$$\langle p \rangle = -\sqrt{2}\, \frac{\hbar}{x_0}\, |\alpha| \, \sin(\omega t - \delta) \ ,$$

where $x_0 = \sqrt{\hbar/m\omega}$ and δ is the phase of α, $\alpha = |\alpha| \exp(i\delta)$.

In other words, the motion of the expectation values is exactly the classical motion and α determines the amplitude and phase of the oscillator.

3. Uncertainty Relation for Coherent States. Calculate $\left\langle \Delta x^2 \right\rangle$ and $\left\langle \Delta p^2 \right\rangle$ for the coherent states introduced in the previous problem and determine the uncertainty product, $\langle \Delta x \rangle \langle \Delta p \rangle$.

4. Harmonic Oscillator in 3D. Consider a 3-dimensional harmonic oscillator with Hamiltonian

$$H = \frac{\boldsymbol{p}^2}{2m} + \frac{m\omega^2 \boldsymbol{x}^2}{2} = \frac{p_x^2}{2m} + \frac{p_y^2}{2m} + \frac{p_z^2}{2m} + \frac{m\omega^2 x^2}{2} + \frac{m\omega^2 y^2}{2} + \frac{m\omega^2 z^2}{2} \,.$$

(a) Show that the energy eigenvalues are $E_n = \hbar\omega(n + 3/2)$.

(b) What is the degeneracy of level n? (That is, how many different states have energy $\hbar\omega(n + 3/2)$?)

5. Scattering by a Square Potential. Consider a plane wave corresponding to a particle of mass m and energy E incident from the left on a potential barrier (or well) of width $2a$ and height V_0 (from $x = -a$ to $x = +a$ to be definite). Calculate the reflection and transmission coefficients. Be sure your results work for positive and negative V_0 and for any $E > 0$.