

1. Some Preliminaries. Assume  $A$  and  $B$  are Hermitian operators.

(a) Show that  $(AB)^\dagger = B^\dagger A^\dagger$ .

(b) Show that  $AB = [A, B]/2 + \{A, B\}/2$  where the *anticommutator*  $\{A, B\} = AB + BA$ . Further, show that the anticommutator is Hermitian and the commutator is anti-Hermitian (that is,  $[A, B]^\dagger = -[A, B]$ ). We know that expectation values of Hermitian operators are real. What can you say about the expectation value of an anti-Hermitian operator?

2. Generalized Uncertainty Principle. Let  $A$  and  $B$  be Hermitian operators. Define the “uncertainty” in  $A$  by the square root of the mean square value:  $\Delta A = \sqrt{\langle (A - \langle A \rangle)^2 \rangle} = \sqrt{\langle A^2 \rangle - \langle A \rangle^2}$ . Show that

$$\Delta A \Delta B \geq \frac{1}{2} | \langle [A, B] \rangle |.$$

Hint: Use the Schwarz Inequality and the results from problem 1. Comment: The commutator of  $x$  and  $p_x$  is  $i\hbar$ , so  $\Delta x \Delta p_x \geq \hbar/2$ .

3. We saw in lecture that the eigenfunction of the momentum operator with eigenvalue  $p$  is  $f_p(x) = (1/\sqrt{2\pi\hbar}) \exp(ipx/\hbar)$ . We are working in one dimension here and we are assuming (almost) the  $\delta$ -function normalization described in lecture. (The extra  $\sqrt{\hbar}$  has been inserted for convenience below.) We also stated that the eigenfunctions of a Hermitian operator form a complete set. This means that for any wave function, we should be able to write

$$\psi(x) = \sum_p c_p f_p(x),$$

where  $c_p$  are the expansion coefficients. Since  $p$  is a continuous variable, the coefficients become a function and we should write:

$$\psi(x) = \int_{-\infty}^{+\infty} \varphi(p) e^{ipx/\hbar} \frac{dp}{\sqrt{2\pi\hbar}},$$

where  $\varphi(p)$  plays the role of the expansion coefficients. Either  $\psi(x)$  or  $\varphi(p)$  is suitable for describing the state of the system. If we are using  $\psi(x)$ , we are using a position space or *configuration space* description. If we are using  $\varphi(p)$ , it's a *momentum space* description.

(a) Show how to determine  $\varphi(p)$  from  $\psi(x)$ . (The above already gave an expression for  $\psi(x)$  in terms of  $\varphi(p)$ .)

(b) What is the meaning of

$$\int_{p_1}^{p_2} |\varphi(p)|^2 dp?$$

(c) What are the position and momentum operators in momentum space? (You can probably make a good guess, but justify your answer!)

4. Projection operators. It's often the case that we want to find the "component" of a function "parallel" to another function. We just take the dot product with the second function, but then we also need to multiply by the second function. A handy notation is

$$|\psi\rangle\langle\psi|.$$

This projects onto  $\psi$ . Operating on  $|\varphi\rangle$ , we get

$$|\psi\rangle\langle\psi|\varphi\rangle,$$

which is what we want! Remember  $\langle\psi|\varphi\rangle$  is just a number and  $|\psi\rangle$  is the vector. Similarly, operating on  $\langle\varphi|$  we get

$$\langle\varphi|\psi\rangle\langle\psi|,$$

which is the desired expression for the adjoint vector. Suppose you have a complete set of orthonormal basis vectors  $|\psi_n\rangle$ . What is a compact expression for transforming an arbitrary vector  $|\varphi\rangle$  into this basis set? (This is much easier to write down than to ask!)

5. We often need to exponentiate a matrix! As an example, let  $A$  be the  $2 \times 2$  matrix

$$A = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

Compute  $e^{itA}$  using two different methods:

(a) Use the Taylor expansion for the exponential:

$$e^{itA} = \sum_{n=0}^{\infty} \frac{(itA)^n}{n!}.$$

(b) Use the spectral decomposition of  $A$ : write  $A = \sum_{k=1}^2 \lambda_k |\psi_k\rangle\langle\psi_k|$  with  $\langle\psi_k|\psi_l\rangle = \delta_{kl}$  and use

$$e^{itA} = \sum_{k=1}^2 e^{it\lambda_k} |\psi_k\rangle\langle\psi_k|.$$