PHYSICS DEPARTMENT, PRINCETON UNIVERSITY

PHYSICS 505 FINAL EXAMINATION

January 13, 2010, 1:30–4:30pm, Jadwin 343

SOLUTIONS

This exam contains five problems. Work any three of the five problems. All problems count equally although some are harder than others. Do all the work you want graded in the separate exam books. *Indicate clearly which three problems you have worked and want graded.* I will only grade three problems. If you hand in more than three problems without indicating which three are to be graded, I will grade the first three, only!

The exam is closed everything: no books, no notes, no calculators, no computers, no cell phones, no ipods, etc.

Write legibly. If I can't read it, it doesn't count!

Put your name on all exam books that you hand in. (Only one should be necessary!!!) On the first exam book, rewrite and sign the honor pledge: *I pledge my honor that I have not violated the Honor Code during this examination.*

If you finish early, do not leave your exam books in the room. Instead, take them to Ms. Angela Glenn in Room 231 Jadwin Hall.

1. Two particles in a box. Two particles of mass m are confined to a rectangular box of sides $a < b < c$. They are in the lowest energy state compatible with the conditions in the cases below. For each of these cases, determine the lowest energy state and its energy and also use first order perturbation theory to determine the correction to the energy if there is an interaction between the particles of the form $V = (V_0(abc))\delta^{(3)}(\boldsymbol{r}_1 - \boldsymbol{r}_2)$.

(a) Two non-identical particles.

Solution

The single particle wave functions will be of the form

$$|n_a, n_b, n_c\rangle = \sqrt{\frac{8}{abc}} \sin \frac{\pi n_a x}{a} \sin \frac{\pi n_b y}{b} \sin \frac{\pi n_c z}{c} \ ,$$

with all quantum numbers greater than 0 and with energy

$$E_{n_a,n_b,n_c} = \frac{\hbar^2 \pi^2}{2m} \left(\left(\frac{n_a}{a}\right)^2 + \left(\frac{n_b}{b}\right)^2 + \left(\frac{n_c}{c}\right)^2 \right) \ .$$

The lowest energy single particle state is $|1, 1, 1\rangle$ with energy

$$E_{1,1,1} = \frac{\hbar^2 \pi^2}{2m} \left(\left(\frac{1}{a}\right)^2 + \left(\frac{1}{b}\right)^2 + \left(\frac{1}{c}\right)^2 \right) \ .$$

Since c is the largest side of the box, the next lowest energy single particle state is $|1, 1, 2\rangle$ with energy

$$E_{1,1,2} = \frac{\hbar^2 \pi^2}{2m} \left(\left(\frac{1}{a}\right)^2 + \left(\frac{1}{b}\right)^2 + \left(\frac{2}{c}\right)^2 \right) \ .$$

We will write two particle states as $|n_a, n_b, n_c\rangle\, |n'_a, n'_b, n'_c\rangle$ where the first ket refers to particle 1 and the second to particle 2.

For the case at hand, non-identical particles, both particles can be placed in the lowest energy single particle state, so the state

$$\psi = |1, 1, 1\rangle\, |1, 1, 1\rangle \ ,$$

with energy

$$E_{1,1,1,1,1,1} = 2E_{1,1,1} = 2\frac{\hbar^2 \pi^2}{2m} \left(\left(\frac{1}{a}\right)^2 + \left(\frac{1}{b}\right)^2 + \left(\frac{1}{c}\right)^2 \right) \ .$$

To take account of the "contact" interaction we need the product of three integrals like

$$\begin{aligned}\Delta_x &= \frac{4}{a^2}\int_0^a dx_1 \int_0^a dx_2 \sin^2\frac{\pi n_a x_1}{a}\sin^2\frac{\pi n_a' x_2}{a}(a\delta(x_1-x_2)) \\ &= \frac{4}{a}\int_0^a dx \sin^2\frac{\pi n_a x}{a}\sin^2\frac{\pi n_a' x}{a} \\ &= 1+\frac{1}{2}\delta_{n_a,n_a'}\end{aligned}$$

For this case, $n_a = n_a' = 1$, and the integral is 3/2. Putting it all together, we find

$$\Delta E = V_0\Delta_x\Delta_y\Delta_z = \frac{27}{8}V_0 \ .$$

End Solution

(b) Two identical particles of spin 0.

Solution

In this case, the wave function must be symmetric under the exchange of the two particles. Since spin 0 particles must necessarily be in a symmetric spin state, the spatial state must be symmetric. The state described in part (a) is symmetric so it works for this part. The state, the energy, and the contact interaction energy are the same as for part (a).

End Solution

(c) Two identical particles of spin 1/2 in the singlet state.

Solution

Now the wave function must be antisymmetric. Since the singlet state is antisymmetric, the spatial state must be symmetric. So the wave function, energy, and contact interaction energy are again, the same as in part (a).

End Solution

(d) Two identical particles of spin 1/2 in the triplet state.

Solution

The wave function must be antisymmetric under exchange of the two particles, the spin state is symmetric, so the spatial state must be antisymmetric. The lowest energy state is

$$\psi = \frac{1}{\sqrt{2}}\left(|1,1,1\rangle\,|1,1,2\rangle - |1,1,2\rangle\,|1,1,1\rangle\right) \ ,$$

with energy

$$E_{1,1,1,1,1,2} = E_{1,1,1}+E_{1,1,2} = \frac{\hbar^2\pi^2}{2m}\left(2\left(\frac{1}{a}\right)^2 + 2\left(\frac{1}{b}\right)^2 + 5\left(\frac{1}{c}\right)^2\right) \ .$$

Now we need the matrix element of the perturbation for this state. The integrals for x and y will be the same as before. The integral for z is,

$$\begin{aligned}
\Delta_z &= \frac{4}{c^2}\frac{1}{2}\int_0^c dz_1 \int_0^c dz_2 \left(\sin\frac{\pi z_1}{c}\sin\frac{2\pi z_2}{c} - \sin\frac{2\pi z_1}{c}\sin\frac{\pi z_2}{c}\right)^2 (c\delta(z_1 - z_2)) \\
&= \frac{2}{\pi}\int_0^\pi dz(\sin z \sin 2z - \sin 2z \sin z) \\
&= 0\,.
\end{aligned}$$

Not surprisingly, the particles in the triplet state are prevented from being right on top of each other by the Pauli exclusion principle, so the contact interaction is zero.

—————————— End Solution ——————————

2. Coupled angular momentum states. A two particle system is in a state $|\psi_0\rangle$, where each particle has orbital angular momentum quantum numbers $l = 1$ and $m_l = 0$. The two particles are **not** identical. The total angular momentum is denoted by $\boldsymbol{J} = \boldsymbol{L}_1 + \boldsymbol{L}_2$ and the eigenvalues of J^2 are $\hbar^2 j(j+1)$.

(a) The two particle state may be expanded in eigenstates of J^2. What values of j have non-zero amplitude in the expansion? For each of these values, what is the probability that it will be found in a measurement of J^2?

Solution

The states of the system may be represented as the direct product of eigenstates of $\boldsymbol{L}_1$ and eigenstates of $\boldsymbol{L}_2$. That is L_1^2 and L_{1z} eigenstates and L_2^2 and L_{2z} eigenstates. Equivalently, they can be represented by eigenstates of J^2, J_z, L_1^2 and L_2^2. The are three possible states for $l = 1$. Since $l_1 = l_2 = 1$, there are nine possible states altogether. Two angular momenta with $l = 1$ can be added to give total angular momenta $j = 2$, five states, $j = 1$, three states, and $j = 0$, one state. Our job is to express total angular momentum states: $|j, m_j, l_1, l_2\rangle$ in terms of products of individual angular momentum states $|l_1, m_1\rangle\,|l_2, m_2\rangle$. Our technique will be to find the highest m_j state in a j multiplet, then successively use the ladder operator to lower m_j until we have all the states in the multiplet. Recall

$$L_\pm\,|l, m\rangle = \sqrt{l(l+1) - m(m\pm 1)}\;|l, m\pm 1\rangle \;.$$

Now, the $|2, 2, 1, 1\rangle$ state requires $m_1 = m_2 = 1$ in order that $m_j = 2$. There is only one way to get this state:

$$|2, 2, 1, 1\rangle = |1, 1\rangle\,|1, 1\rangle \;,$$

where first ket on the right hand side refers to particle 1 and the second to particle two. Operating with $J_- = L_{1-} + L_{2-}$, we have

$$\begin{aligned} J_-\,|2, 2, 1, 1\rangle &= 2\,|2, 1, 1, 1\rangle \\ &= L_{2-}\,|1, 1\rangle\,|1, 1\rangle + L_{1-}\,|1, 1\rangle\,|1, 1\rangle \;, \\ &= \sqrt{2}\,|1, 1\rangle\,|1, 0\rangle + \sqrt{2}\,|1, 0\rangle\,|1, 1\rangle \end{aligned}$$

or

$$|2, 1, 1, 1\rangle = \frac{1}{\sqrt{2}}(|1, 1\rangle\,|1, 0\rangle + |1, 0\rangle\,|1, 1\rangle) \,.$$

Applying J_- three more times, we find

$$|2, 0, 1, 1\rangle = \frac{1}{\sqrt{6}}\left(|1, 1\rangle\,|1, -1\rangle + 2\,|1, 0\rangle\,|1, 0\rangle + |1, -1\rangle\,|1, 1\rangle\right) ,$$

$$|2, -1, 1, 1\rangle = \frac{1}{\sqrt{2}}(|1, 0\rangle\,|1, -1\rangle + |1, -1\rangle\,|1, 0\rangle) \,,$$

$$|2, -2, 1, 1\rangle = |1, -1\rangle\,|1, -1\rangle \;.$$

The state $|1,1,1,1\rangle$ must be a superposition of the same states that make up $|2,1,1,1\rangle$ in order to get the correct J_z. In addition $|1,1,1,1\rangle$ and $|2,1,1,1\rangle$ must be orthogonal. This determines $|1,1,1,1\rangle$ up to a phase:

$$|1,1,1,1\rangle = \frac{1}{\sqrt{2}}(|1,1\rangle\,|1,0\rangle - |1,0\rangle\,|1,1\rangle)\,.$$

Applying J_- twice, we find

$$|1,0,1,1\rangle = \frac{1}{\sqrt{2}}\left(|1,1\rangle\,|1,-1\rangle - |1,-1\rangle\,|1,1\rangle\right),$$

$$|1,-1,1,1\rangle = \frac{1}{\sqrt{2}}(|1,0\rangle\,|1,-1\rangle - |1,-1\rangle\,|1,0\rangle)\,,$$

Finally, $|0,0,1,1\rangle$ must be a superposition of the same states that make up $|1,0,1,1\rangle$ and $|2,0,1,1\rangle$ and it must be orthogonal to both. This determines (up to a phase)

$$|0,0,1,1\rangle = \frac{1}{\sqrt{3}}\left(|1,1\rangle\,|1,-1\rangle - |1,0\rangle\,|1,0\rangle + |1,-1\rangle\,|1,1\rangle\right).$$

Now we know the expansions of $|j,m_j,1,1\rangle$ states in terms of $|1,m_1\rangle\,|1,m_2\rangle$ states, but what we really want is the expansion of $|1,0\rangle\,|1,0\rangle$ in terms of $|j,0,1,1\rangle$ states. There are three such states:

$$|2,0,1,1\rangle = \frac{1}{\sqrt{6}}\left(|1,1\rangle\,|1,-1\rangle + 2\,|1,0\rangle\,|1,0\rangle + |1,-1\rangle\,|1,1\rangle\right),$$

$$|1,0,1,1\rangle = \frac{1}{\sqrt{2}}\left(|1,1\rangle\,|1,-1\rangle - |1,-1\rangle\,|1,1\rangle\right),$$

$$|0,0,1,1\rangle = \frac{1}{\sqrt{3}}\left(|1,1\rangle\,|1,-1\rangle - |1,0\rangle\,|1,0\rangle + |1,-1\rangle\,|1,1\rangle\right).$$

Solving, we find

$$|1,0\rangle\,|1,0\rangle = \frac{1}{\sqrt{3}}(\sqrt{2}\,|2,0,1,1\rangle - |0,0,1,1\rangle)\,.$$

We might have guessed that $|2,0,1,1\rangle$ and $|0,0,1,1\rangle$ would be the only states present because these have even parity as does $|0,1\rangle\,|0,1\rangle$ which is the product of two odd parity states, while $|1,0,1,1\rangle$ has odd parity. However, to know the probabilities, we have to work out the expansion. When measuring J^2, the probability of getting $j=2$ is 2/3, the probability of getting $j=0$ is 1/3 and the probability of getting $j=1$ is 0.

End Solution

At $t = 0$, a coupling between the angular momenta is turned on so the system has Hamiltonian,

$$H = \gamma \boldsymbol{L}_1 \cdot \boldsymbol{L}_2 \,,$$

where γ is a constant. The state of the system now depends on time, $|\psi(t)\rangle$.

(b) Show that $|\langle\psi(t) \mid \psi_0\rangle|^2$ is periodic with period T and find T. Also evaluate $|\langle\psi(t) \mid \psi_0\rangle|^2$ at $t = T/2$

Solution

$$H = \gamma \boldsymbol{L}_1 \cdot \boldsymbol{L}_2 = \frac{\gamma}{2}(J^2 - L_1^2 - L_2^2) \,,$$

so the states $|j, m_j, l_1, l_2\rangle$ are eigenstates of the new Hamiltonian, whereas the states $|l_1, m_1\rangle\,|l_2, m_2\rangle$ are not. The expansion of the initial state in the states $|j, m_j, l_1, l_2\rangle$ was worked out in part (a). The time evolution is found by multiplying each stationary state by $\exp -iEt/\hbar$ where E is the energy of the stationary state. For $j = 2$, the expression above gives $E_2 = \gamma\hbar^2$ and for $j = 0$, $E_0 = -2\gamma\hbar^2$. So,

$$|\psi(t)\rangle = \sqrt{\frac{2}{3}}\;|2,0,1,1\rangle\; e^{-i\gamma\hbar t} - \sqrt{\frac{1}{3}}\;|0,0,1,1\rangle\; e^{+2\gamma\hbar t} \,,$$

and

$$\langle\psi(t) \mid \psi_0\rangle = \frac{2}{3}\, e^{+i\gamma\hbar t} + \frac{1}{3}\, e^{-2i\gamma\hbar t} = e^{i\gamma\hbar t}\left(\frac{2}{3} + \frac{1}{3}\, e^{-3i\gamma\hbar t}\right) .$$

Then

$$|\langle\psi(t) \mid \psi_0\rangle|^2 = \frac{5}{9} + \frac{4}{9}\,\cos(3\gamma\hbar t) \,,$$

so indeed, the probability of finding the system in the same state as $|\psi_0\rangle$ is periodic and has period

$$T = \frac{2\pi}{3\gamma\hbar} \,.$$

For $t = T/2$,

$$|\langle\psi(t) \mid \psi_0\rangle|^2 = \frac{5}{9} + \frac{4}{9}\,\cos(\pi) = \frac{1}{9} \,,$$

which means the state is mostly $|1,1\rangle\,|1-1\rangle$ and $|1,-1\rangle\,|1,1\rangle$.

End Solution

3. Electric pulse. Consider hydrogen in its ground state for $t \le 0$. It's acted on by an electric field pulse in the z-direction of the form

$$\boldsymbol{E}(t) = \begin{cases} 0 & \text{if } t \le 0 \\ E_0 \boldsymbol{e}_z e^{-t/\tau} & \text{if } t > 0 \end{cases} ,$$

where E_0 and τ are positive constants. Such a field can be produced by placing the atom between the plates of a capacitor and pulsing the capacitor. Obtain first order expressions for the probability that the atom winds up in a 2s state and the probability that the atom winds up in a 2p state at $t = +\infty$.

Useful data: for hydrogen,

$$R_{10}(r) = 2\left(\frac{1}{a}\right)^{3/2} e^{-r/a} ,$$

$$R_{20}(r) = 2\left(\frac{1}{2a}\right)^{3/2} \left(1 - \frac{r}{2a}\right) e^{-r/2a} ,$$

$$R_{21}(r) = \frac{1}{\sqrt{3}}\left(\frac{1}{2a}\right)^{3/2} \frac{r}{a} e^{-r/2a} ,$$

$$a = \frac{\hbar^2}{me^2} .$$

Solution

The amplitude to go from state $|0\rangle$ to state $|f\rangle$ is

$$\langle f, +\infty \mid 0, 0\rangle = \frac{1}{i\hbar}\int_0^{+\infty} dt\, e^{i\omega_{f0} t} \langle f \mid V(t) \mid 0\rangle ,$$

where $\omega_{f0} = (E_f - E_0)/\hbar$ and where the perturbing potential is $V(t) = +eE(t)z$. (The potential can be thought of as acting on the electron.) Now, $z = r\cos\theta$. The $2s$ state is even in z as is the $1s$ ground state. So there is no amplitude to go to the $2s$ state. (Basically, the same selection rule that requires l to change by one unit in a radiative transition!) The $2p$ states involve $Y_{1,+1}$, $Y_{1,0}$, and $Y_{1,-1}$. For the $m = \pm 1$ states, the integral over ϕ will give 0. So, we need to evaluate

$$\int \sin\theta\, d\theta\, d\phi\, Y^*_{1,0}(\theta, \phi) \cos\theta\, Y_{0,0}(\theta, \phi) = \frac{1}{\sqrt{3}} .$$

If you forgot the normalization of $Y_{1,0}$, it's easy to work out. The radial part requires the evaluation of

$$\int_0^\infty r^2\, dr\, R_{21}(r)\, r\, R_{1,0}(r) = \frac{2^{15/2} a}{3^{9/2}} .$$

Next we need to evaluate the time integral which is really the Fourier transform of the potential at the frequency corresponding to the energy difference.

$$\int_0^{+\infty} dt\, e^{i\omega_{f0}t} eE_0 e^{-t/\tau} = \left. \frac{eE_0 e^{-t/\tau + i\omega_{f0}t}}{-1/\tau + i\omega_{f0}} \right|_0^{+\infty} = \frac{eE_0}{1/\tau - i\omega_{f0}} \, .$$

The frequency is

$$\omega_{f0} = \frac{E_f - E_0}{\hbar} = \left(-\frac{1}{4}\frac{e^2}{2a} + \frac{e^2}{2a} \right) \Big/ \hbar = \frac{3e^2}{8\hbar a} \, .$$

We multiply everything together to get the transition amplitude,

$$\langle f, +\infty \mid 0, 0 \rangle = \frac{2^{15/2}}{3^5 i} \frac{eE_0 a\tau}{\hbar(1 - i\omega_{f0}\tau)} \, ,$$

and the transition probability is

$$P_{0\to 1} = \frac{2^{15}}{3^{10}} \frac{(eE_0 a\tau)^2}{\hbar^2(1 + \omega_{f0}^2\tau^2)} \, ,$$

with $\omega_{f0} = 3e^2/(8\hbar a)$.

——————————— End Solution ———————————

4. Particle in a box. A particle of mass m is confined to the cubical box $0 \le x \le L$, $0 \le y \le L$, $0 \le z \le L$ by a potential that is 0 inside the box and very large (∞) outside the box.

(a) What are the normalized wavefunctions and energies of the stationary states of the particle in this box? Be sure to give the quantum numbers necessary to specify the state.

Solution

The wave functions are just products of sine waves that must be vanish at 0 and L along each coordinate:

$$\psi_{pqr}(x, y, z) = \left(\frac{2}{L}\right)^{3/2} \sin\left(\frac{\pi p x}{L}\right) \sin\left(\frac{\pi q y}{L}\right) \sin\left(\frac{\pi r z}{L}\right) ,$$

where the quantum numbers $p \ge 1$, $q \ge$, and $r \ge 1$ are integers in order that the boundary conditions at the edge of the box are satisfied. The energies are

$$E_{pqr} = \frac{\hbar^2 \pi^2}{2mL^2}(p^2 + q^2 + r^2) .$$

End Solution

(b) Consider an energy E such that $E \gg E_0$, where E_0 is the ground state energy. What is the density of states (number of states per unit energy) at energy $E + E_0 \approx E$?

Solution

The number of states with energy less than E is the number of triplets of integers, p, q, r with

$$\frac{\hbar^2 \pi^2}{2mL^2}(p^2 + q^2 + r^2) < E ,$$

or

$$(p^2 + q^2 + r^2) < \frac{2mL^2}{\hbar^2 \pi^2} E ,$$

The number of such triplets is the number of lattice points in the eighth of a sphere in p, q, r space of radius

$$R < \left(\frac{2mL^2}{\hbar^2 \pi^2} E\right)^{1/2} ,$$

but this is just the volume of the octant

$$N(< E) = \frac{\pi}{6} \left(\frac{2mL^2}{\hbar^2 \pi^2}\right)^{3/2} E^{3/2} .$$

The density of states at E is

$$\rho(E) = \frac{dN}{dE} = \frac{\pi}{4} \left(\frac{2mL^2}{\hbar^2 \pi^2}\right)^{3/2} E^{1/2} .$$

This result can be obtained another way. Consider wave number space (k_x, k_y, k_z). The density of states in wave number space is $(L/\pi)^3$ so the number of states in the shell $(4\pi/8)k^2\,dk$ is

$$\rho(k)\,dk = \frac{\pi}{2}\left(\frac{L}{\pi}\right)^3 k^2\,dk\;,$$

where the factor of 1/8 accounts for the fact that we are only counting states with all components of k greater than 0. Converting from k to $E = \hbar^2 k^2/2m$, $dE = \hbar^2 k\,dk/m$ we have

$$\rho(E)\,dE = \frac{\pi}{4}\left(\frac{L}{\pi}\right)^3\left(\frac{2m}{\hbar^2}\right)^{3/2} E^{1/2}\,dE\;,$$

the same as before.

End Solution

(c) At $t = 0$, with the particle in its ground state, a weak, time dependent, potential is turned on:

$$V(\boldsymbol{x}, t) = V_0 L^3\,\delta(x - L/2)\,\delta(y - L/2)\,\delta(z - L/2)\,\cos(\omega t)\;,$$

where $\hbar\omega \gg E_0$. This potential is an oscillating "spike" at the center of the box. It can cause transitions from the ground state to excited states. What are the selection rules for these transitions? That is, what constraints must the quantum numbers of the excited states satisfy?

Solution

The transition rate involves the matrix element of the potential between the ground state and the excited state. The potential is non-zero only at $(L/2, L/2, L/2)$, the center of the box. Any wave function in which one or more of the quantum numbers is even vanishes at the center of the box. Therefore, the selection rule is that all quantum numbers must be odd! Note that 1/8 of the states have quantum numbers which are all odd (and 1/8 all even, and 3/8 two evens and one odd and 3/8 two odds and one even).

End Solution

(d) What is the first order transition rate, from the ground state to the total of all excited states allowed by energy conservation and the selection rules, produced by the oscillating potential of part (c)?

Solution

For this part, we use Fermi's Golden Rule:

$$\Gamma = \left(\frac{2\pi}{\hbar}\right)\,|\,\langle f\mid V\mid 0\rangle\,|^2\,\rho(E_f)\;,$$

where $\langle f|$ denotes a final state, V is the spatial part of the oscillating potential (whatever multiples $e^{i\omega t}$), $|0\rangle$ denotes the ground state, and $E_f \approx \hbar\omega$ is the energy of the final state.

If you didn't remember the golden rule, it can be gotten by dimensional analysis. First everything depends on the matrix element of the perturbation squared. This has dimensions of energy squared. It should be proportional to the number of available final states, which is proportional to the density of states which has the dimensions of inverse energy. A rate is inverse time, so if we divide by $\hbar$ we wind up with inverse time. All the factors are accounted for except the 2π.

The matrix element is

$$\left\langle pqr \mid (V_0/2)\, L^3\, \delta(x - L/2)\, \delta(y - L/2)\, \delta(z - L/2) \mid 111 \right\rangle = \pm 4V_0\ ,$$

provided p, q, and r satisfy the selection rules and are all odd. Note that a wave function with all odd quantum numbers is $\pm(2/L)^{3/2}$ at the center of the box, so the integral is very easy to do. Also, the factor of 1/2 comes from the 1/2 in the definition of the cosine: $(e^{+i\omega t} + e^{-i\omega t})/2$. For the density of states, we use 1/8 that calculated in part (b) since only 1/8 of the states satisfy the selection rule. Putting everything together:

$$\Gamma = \frac{2\pi}{\hbar}\, 16V_0^2\, \frac{1}{8}\, \frac{\pi}{4} \left(\frac{2mL^2}{\hbar^2\pi^2}\right)^{3/2} (\hbar\omega)^{1/2} = \frac{2}{\pi}\, \frac{mV_0^2 L^3}{\hbar^3} \left(\frac{2m\omega}{\hbar}\right)^{1/2}\ .$$

——— End Solution ———

5. Scattering from a spherical shell. A particle of mass m and energy $E = \hbar^2 k^2/2m$ is scattered by the fixed, spherically symmetric potential

$$V(r) = -V_0 a \delta(r - a) ,$$

where V_0 and a are a positive constants. In the following, use suitable approximations.

(a) What is the total scattering cross section at very low energies ($ka \ll 1$)?

Solution

At low energies, the scattering is spherically symmetric and only the s-wave scattering is important. In this case all we need to do is calculate the s-wave phase shift, δ_0. The total cross section is

$$\sigma = \frac{4\pi}{k^2} \sin^2 \delta_0 .$$

So the problem is to calculate δ_0. Writing the Schroedinger equation in spherical coordinates, the angular parts vanish for $l = 0$. If we let $u(r) = rR(r)$, where $R(r)$ is the radial wave function, the Schroedinger equation becomes

$$\frac{d^2u}{dr^2} - \frac{2m}{\hbar^2} V u + k^2 u = 0 .$$

The solution must vanish at $r = 0$, so for $r < a$ the solution is

$$u_i(r) = \sin kr .$$

For $r > a$, the solution must be of the form of an incoming and an outgoing wave with a phase shift for the outgoing wave. Also, the if there were no scattering, the incoming and outgoing waves must add up to a function proportional to $\sin kr$. So, the solution for $r > a$ looks like

$$\begin{aligned} u(r) &= -e^{-ikr} + e^{ikr + 2i\delta_0} \\ &= \left(-e^{-ikr - i\delta_0} + e^{ikr + i\delta_0}\right) e^{i\delta_0} \\ &= \sin(kr + \delta_0)\left(2ie^{i\delta_0}\right) . \end{aligned}$$

We can absorb the last factor in the overall normalization constant and the solution for $r > a$ can be written

$$u_o(r) = A \sin(kr + \delta_0) ,$$

where A is the normalization constant and δ_0 is the s-wave phase shift we want to find. The wave function must be continuous at $r = a$ which means

$$\sin(ka) = A \sin(ka + \delta_0) ,$$

or

$$A = \frac{\sin(ka)}{\sin(ka + \delta_0)} .$$

Since we have a δ-function potential, there is a discontinuity in the slope of the wave function at $r = a$. We find this discontinuity by integrating the Schroedinger equation from $a - \epsilon$ to $a + \epsilon$ and taking the limit as $\epsilon \to 0$. The first term gives

$$\begin{aligned} u'_o - u'_i &= Ak\cos(ka + \delta_0) - k\cos(ka) \\ &= \frac{k}{\sin(ka+\delta_0)}\left(\sin(ka)\cos(ka+\delta_0) - \sin(ka+\delta_0)\cos(ka)\right) \\ &= -\frac{k\sin\delta_0}{\sin(ka+\delta_0)} \; . \end{aligned}$$

The second term gives

$$-\int_{a-\epsilon}^{a+\epsilon} \frac{2m}{\hbar^2} V(r)u(r)dr = +\frac{2mV_0a}{\hbar^2}\sin(ka) \; .$$

The third term in the Schroedinger equaton gives zero. Putting it all together, we have

$$\frac{k\sin\delta_0}{\sin(ka+\delta_0)} = \frac{2mV_0a}{\hbar^2}\sin(ka) \; .$$

This is a transcendental equation for δ_0, so in principle, we've solved the problem. However, we can simplify it a bit more. Since we are assuming $ka \ll 1$, on the right hand side we can replace $\sin(ka)$ by ka. We let $2mV_0a^2\hbar^2 = (V_0/E)(ka)^2 = C$. Then

$$\sin\delta_0 = C\sin(ka+\delta_0) = C(ka)\cos\delta_0 + C\sin\delta_0 \; .$$

Or

$$\begin{aligned} \sin\delta_0(1-C) &= C(ka)\cos\delta_0 \; , \\ \sin^2\delta_0\,(1 - 2C + C^2) &= C^2(ka)^2 - C^2(ka)^2\sin^2\delta_0 \; , \\ \sin^2\delta_0 &= \frac{C^2(ka)^2}{1 - 2C + C^2(1 + (ka)^2)} \; . \end{aligned}$$

Finally,

$$\sigma = \frac{4\pi}{k^2}\sin^2\delta_0 = \frac{4\pi C^2a^2}{1 - 2C + C^2(1 + (ka)^2)} \; ,$$

with $C = (V_0/E)(ka)^2$. Note that if C is very small, the cross section goes to $4\pi C^2a^2$ and if C is very large, the cross section becomes $4\pi a^2$. The extrema of the expression occur when $C = 0$, where the cross section is 0 and $C = 1$ where the cross section is $4\pi/k^2$.

——— End Solution ———

(b) What is the differential cross section at very high energies ($ka \gg 1$)?

——— Solution ———

For high energies, the Born approximation is suitable. The scattering amplitude is

$$f(\theta,\phi) = -\frac{m}{2\pi\hbar^2}\int V(\boldsymbol{r}')e^{-i\Delta\boldsymbol{k}\cdot\boldsymbol{r}'}\,d^3r'\,.$$

Since the potential is spherically symmetric, there will be no ϕ dependence and we can take $\phi = 0$, $k_i = (0,0,k)$, $k_o = (k\sin\theta, 0, k\cos\theta)$. Then the magnitude of $\Delta\boldsymbol{k}$ is $2k\sin\theta/2$ and it points in the xz-plane. In order to do the integral, take the z'-direction to be along $\Delta\boldsymbol{k}$. The ϕ' integral produces 2π and we are left with

$$\begin{aligned}
f(\theta) &= -\frac{2\pi m}{2\pi\hbar^2}\int_0^\infty r'^2\,dr'\int_0^\pi \sin\theta'\,d\theta'\,(-V_0 a)\delta(r'-a)e^{-2ik\sin(\theta/2)r'\cos\theta'}\\
&= \frac{2\pi m V_0 a^3}{2\pi\hbar^2}\int_0^\pi \sin\theta'\,d\theta'\,e^{-2ik\sin(\theta/2)a\cos\theta'}\\
&= \frac{2\pi m V_0 a^2}{4\pi i\hbar^2 k\sin(\theta/2)}\,e^{-2ik\sin(\theta/2)a\cos\theta'}\Big|_0^\pi\\
&= \frac{2mV_0a^3}{\hbar^2}\frac{\sin(2ka\sin\theta/2)}{2ka\sin\theta/2}\,.
\end{aligned}$$

The differential scattering cross section is thus

$$\frac{d\sigma}{d\Omega} = \left(\frac{2mV_0a^3}{\hbar^2}\right)^2\left(\frac{\sin(2ka\sin\theta/2)}{2ka\sin\theta/2}\right)^2\,.$$

To get the total cross section, we must integrate the differential cross section over solid angle.

——————— End Solution ———————