Problem set 11 solutions

Problem 1.

part a. In the relaxation time approximation, the Boltzmann equation is
\[
\frac{\partial f}{\partial t} + \vec{v} \cdot \nabla f + \vec{a} \cdot \nabla_v f = -\frac{f - f_0}{\tau_c}.
\]

We consider a steady state configuration, with \( \frac{\partial f}{\partial t} = 0 \). Since there is no external potential, the acceleration \( \vec{a} = 0 \). \( f_0 \) is dependent on \( x \), and \( f_1 \) is assumed to be much less than \( f_0 \), so we can neglect the contribution of \( f_1 \) to the gradient term on the left hand side. Thus, using \( f = f_0 + f_1 \), we’re left with
\[
-\frac{f_1}{\tau_c} = v_x \frac{\partial f_0}{\partial x} = v_x \frac{\partial f_0}{\partial \tau} \frac{d\tau}{dx}.
\]

The equilibrium distribution function \( f_0 = \exp \left( \frac{\mu - \epsilon}{\tau} \right) \) is a function of energy \( \epsilon(\vec{v}) \), chemical potential \( \mu(x) = \tau(x) \log(n/n_Q(x)) \), and temperature \( \tau(x) \). Differentiating with respect to temperature gives
\[
\frac{\partial f_0}{\partial \tau} = \exp \left( \frac{\mu - \epsilon}{\tau} \right) \frac{\partial}{\partial \tau} \left( \frac{\mu - \epsilon}{\tau} \right) = f_0 \left( -\frac{3}{2\tau} + \frac{\epsilon}{\tau^2} \right).
\]

Plugging this into the expression for \( f_1 \) gives
\[
f = f_0 + f_1 = f_0 - \tau_c v_x f_0 \frac{d\tau}{dx} \left( -\frac{3}{2\tau} + \frac{\epsilon}{\tau^2} \right).
\]

part b. The energy flux in the \( x \) direction is given by
\[
J_x = \int v_x(\epsilon) f \mathcal{D}(\epsilon) d\epsilon.
\]

Since there is no net energy flux at equilibrium, the zeroth order term in \( J_x \) is zero, and we may replace \( f \) by \( f_1 \) in the above expression. Plugging in \( f_1 = f - f_0 \) from part a gives
\[
J_x = -\tau_c \frac{d\tau}{dx} \int v_x^2 f_0 \left( -\frac{3}{2\tau} + \frac{\epsilon}{\tau^2} \right) \mathcal{D}(\epsilon) d\epsilon.
\]

part c.
\[
J_x = -\tau_c \frac{d\tau}{dx} \int v_x^2 \exp \left( \frac{\mu - \epsilon}{\tau} \right) \left( -\frac{3}{2\tau} + \frac{\epsilon}{\tau^2} \right) \mathcal{D}(\epsilon) d\epsilon
\]
\[
= -\tau_c \frac{d\tau}{dx} e^{\mu/\epsilon} \left( \frac{2\epsilon}{3m} e^{-\epsilon/\tau} \left( -\frac{3}{2\tau} + \frac{\epsilon}{\tau^2} \right) \left( \frac{1}{4\pi^2} \frac{2m}{\hbar^2} \right)^{3/2} \epsilon^{1/2} \right) d\epsilon
\]
\[
= -\tau_c e^{\mu/\epsilon} \frac{d\tau}{dx} \left( \frac{2}{3m} \frac{1}{4\pi^2} \frac{2m}{\hbar^2} \right)^{3/2} \left( -\frac{3}{2} \left( \frac{7}{2} \right) + \Gamma \left( \frac{9}{2} \right) \right)
\]
\[
= -\frac{5n\tau_c}{m} \frac{d\tau}{dx}
\]
\[
\Rightarrow K = \frac{5n\tau_c}{m}
\]
Problem 2. Selenium is the only non-metallic element that is solid at room temperature and has a sound speed listed on http://www.webelements.com. The relevant physical quantities for selenium are \( c = 3350 \text{ m/s}, K = 0.52 \text{ W m}^{-1} \text{ K}^{-1}, \) \( C_p = 25.36 \text{ J mol}^{-1} \text{ K}^{-1}, n = 6.09 \times 10^4 \text{ mol m}^{-3}, \) and \( \ell \approx n^{-1/3} = 3.01 \times 10^{-10} \text{ m}. \) Computing the thermal diffusivity using the sound speed and lattice spacing gives \( D = \frac{1}{c} \ell^2 = 3.36 \times 10^{-7} \text{ m}^2 \text{ s}^{-1}. \) and computing \( D \) using the thermal conductivity gives \( D = \frac{K}{\kappa_m} = 3.37 \times 10^{-7} \text{ m}^2 \text{ s}^{-1}. \) These values are remarkably similar.

Problem 3. The solution to the heat equation in a semi-infinite medium is given in equation 15.13 in the text, so we need only combine solutions of the appropriate frequencies. The temperature as a function of depth \( z \) and time \( t \) is
\[
\theta(z, t) = \theta_0 + \theta_d e^{-z/\delta_d} \cos(\omega_d t - z/\delta_d) + \theta_a e^{-z/\delta_a} \cos(\omega_a t - z/\delta_a),
\]
where the subscripts “d” and “a” denote “daily” and “annual”, respectively. When the daily and the annual fluctuations are each at a minimum, the temperature will be \( \theta_{\text{min}}(z) = \theta_0 - \theta_d e^{-z/\delta_d} - \theta_a e^{-z/\delta_a}. \) In order to keep the pipes from freezing, they must be buried deeply enough that the minimum temperature (in degrees Celsius) is nonnegative. Setting \( \theta_{\text{min}} = 0, \) and using the definition of \( \delta \) in the text, we can solve numerically to find \( z = 11.6 \text{ cm}. \)

Problem 4. Assume that the slab is infinite in the \( x \) and \( y \) directions, so that the temperature depends only on \( z \) and \( t. \) We begin by writing the temperature as a sum of plane waves,
\[
\tau(z, t) = \theta_0 + \sum_{n=1}^{\infty} \tau_n \exp(i k_n z - i \omega_n t).
\]
At all \( t, \) \( \tau(0, t) - \theta_0 = 0. \) Thus, setting \( t = 0 \) and \( z = 0, \) we have \( \Re(\tau_n) = 0 \) for all \( n. \) Similarly, at \( t = 0 \) and \( z = 2a, \) \( \theta(2a, 0) - \theta_0 = 0 = \Re(\tau_n \exp(2i n a)) = \Im(\tau_n) \sin(2n \pi a) = 0, \) which implies \( k_n = \frac{n \pi}{2a}. \) This implies that, at \( t = 0, \)
\[
\tau(z, 0) - \theta_0 = \theta_1 - \theta_0 = \sum_{n=1}^{\infty} \Im(\tau_n) \sin\left(\frac{n \pi z}{2a}\right).
\]
Multiplying by \( \sin\left(\frac{m \pi z}{2a}\right), \) and integrating with respect to \( z \) from 0 to \( 2a, \) gives the coefficients \( \Im(\tau_m), \)
\[
\int_0^{2a} (\theta_1 - \theta_0) \sin\left(\frac{m \pi z}{2a}\right) dz = \sum_{n=1}^{\infty} \Im(\tau_n) \int_0^{2a} \sin\left(\frac{n \pi z}{2a}\right) \sin\left(\frac{m \pi z}{2a}\right) dz
\]
\[
= \sum_{n=1}^{\infty} \Im(\tau_n) a \delta_{mn} = \Im(\tau_m) a
\]
\[
\Rightarrow \Im(\tau_m) = \frac{1}{a} \int_0^{2a} (\theta_1 - \theta_0) \sin\left(\frac{m \pi z}{2a}\right) dz
\]
\[
= \frac{4(\theta_1 - \theta_0)}{m \pi} \text{ for odd } m, \text{ and } 0 \text{ for even } m.\]
Finally, using the dispersion relation $Dk^2 = i\omega$, we can write down the temperature as a function of position and time,

$$\tau(z,t) = \theta_0 + \sum_{n=0}^{\infty} \frac{4(\theta_1 - \theta_0)}{(2n + 1)\pi} \sin \left( \frac{(2n + 1)\pi z}{2a} \right) \exp \left( -\frac{Dn^2\pi^2 t}{4a^2} \right).$$

At large $t$, $\tau(z,t) \approx \theta_0 + \frac{4(\theta_1 - \theta_0)}{\pi} \sin \left( \frac{\pi z}{2a} \right) \exp \left( -\frac{D\pi^2 t}{4a^2} \right)$. The value of $t$ at which $\tau(a,t) - \theta_0 = (\theta_1 - \theta_0)/100$ is given by

$$\frac{\theta_1 - \theta_0}{100} = \tau(a,t) - \theta_0 = \frac{4(\theta_1 - \theta_0)}{\pi} \sin(\pi/2) \exp \left( -\frac{D\pi^2 t}{4a^2} \right).$$

$$\Rightarrow \exp \left( -\frac{D\pi^2 t}{4a^2} \right) = \frac{\pi}{400} \Rightarrow t = \frac{4a^2}{D\pi^2} \log \left( \frac{400}{\pi} \right).$$

**Problem 5.**

**part a.** Putting $\hat{J}_u = -K\vec{\nabla}\tau$ into $\hat{C}\frac{\partial r}{\partial t} + \vec{\nabla} \cdot \vec{J}_u = g_u$ gives $\hat{C}\frac{\partial r}{\partial t} = K\nabla^2 \tau + g_u$. At steady state, $\frac{\partial \tau}{\partial t} = 0$, and we have $\nabla^2 \tau = -g_u/K$. Since $\tau$ is independent of $z$ and $\varphi$, the Laplacian of $\tau$ in cylindrical coordinates is $\nabla^2 \tau = \frac{1}{r} \frac{\partial}{\partial r} \left( r \frac{\partial \tau}{\partial r} \right)$, so

$$\frac{\partial}{\partial r} \left( r \frac{\partial \tau}{\partial r} \right) = -\frac{g_u r}{K} \Rightarrow r^2 \frac{\partial \tau}{\partial r} = -\frac{g_u r^2}{2K} + \text{const.}$$

In order for $\tau$ to be finite at $r = 0$, we must have $\lim_{r \to 0} r \frac{\partial \tau}{\partial r} = 0$, so the constant above vanishes. Integrating once more, we find $\tau(r) = -\frac{g_u r^2}{4K} + \text{const}$. The temperature difference between the outer surface $r = a$ and the center is

$$\Delta \tau = \tau(0) - \tau(a) = \frac{g_u a^2}{4K}.$$

**part b.** For a spherical geometry,

$$\nabla^2 \tau = \frac{1}{r^2} \frac{\partial}{\partial r} \left( r^2 \frac{\partial \tau}{\partial r} \right) = -\frac{g_u}{K}$$

$$\Rightarrow \frac{\partial}{\partial r} \left( r^2 \frac{\partial \tau}{\partial r} \right) = -\frac{g_u r^2}{K}$$

$$\Rightarrow r^2 \frac{\partial \tau}{\partial r} = -\frac{g_u r^3}{3K}$$

$$\Rightarrow \tau(r) = -\frac{g_u r^3}{6K} + \text{const}$$

$$\Rightarrow \Delta \tau = \tau(0) - \tau(a) = \frac{g_u a^2}{6K}.$$