Problem 1.

part a.

\[ U = \frac{3}{2} N\tau - \frac{N^2 a}{V} \]

\[ \Rightarrow H = U + pV = \frac{3}{2} N\tau - \frac{N^2 a}{V} + \frac{N\tau V}{V - Nb} - \frac{N^2 a}{V} \]

\[ = \frac{3}{2} N\tau + N\tau \left( \frac{1}{1 - Nb/V} \right) - \frac{2N^2 a}{V} \]

\[ = N\tau \left( \frac{3}{2} + \frac{1}{1 - Nb/V} \right) - \frac{2N^2 a}{V} \]

\[ \lambda = \frac{H_{out} - H_{in}}{H_{vap} - H_{liq}} \]  

The denominator of this expression is the enthalpy change associated with the transition from a liquid at \( \tau_{liq} \) and \( p_{out} \) to a gas at \( \tau_{in} \) and \( p_{out} \). In order to make this transition, the liquid would first have to be evaporated, involving an enthalpy change \( H_{vap} \), and then would have to be heated to a temperature \( \tau_{in} \), for which \( \Delta H = C_p \Delta\tau = \frac{5}{2} N (\tau_{in} - \tau_{liq}) \).

In order to compute the numerator in the expression for \( \lambda \), we use the formula for enthalpy derived above.

\[ H_{out} - H_{in} = N \left( \frac{\tau_{in}}{1 - Nb/V_{out}} - \frac{\tau_{in}}{1 - Nb/V_{in}} - \frac{2Na}{V_{out}} + \frac{2Na}{V_{in}} \right) \]

approximating \( N \frac{V}{V} \approx \frac{p}{\tau} \) gives

\[ H_{out} - H_{in} \approx N \left( \frac{\tau_{in}}{1 - p_{out}b/\tau_{in}} - \frac{\tau_{in}}{1 - p_{in}b/\tau_{in}} - \frac{2ap_{out}}{\tau} + \frac{2ap_{in}}{\tau} \right) \]

\[ = N \left( \frac{\tau_{in}^2}{\tau_{in} - p_{out}b} - \frac{\tau_{in}^2}{\tau_{in} - p_{in}b} - \frac{2a(p_{out} - p_{in})}{\tau_{in}} \right) \]

The constant \( b \) can be computed from the volume per atom,

\[ b = \frac{\text{volume per atom}}{2} = \frac{32.0 \text{cm}^3}{2 \times 6.02 \times 10^{23}} = 2.66 	imes 10^{-29} \text{m}^3 \]

Now that \( b \) is known, \( a \) can be computed from the inversion temperature,

\[ a = \frac{b\tau_{in}}{2} = 9.35 \times 10^{-51} \text{J m}^3 \]

Finally, we have an expression for \( \lambda \),

\[ \lambda \approx \frac{N \left( \frac{\tau_{in}^2}{\tau_{in} - p_{out}b} - \frac{\tau_{in}^2}{\tau_{in} - p_{in}b} - \frac{2a(p_{out} - p_{in})}{\tau_{in}} \right)}{\Delta H_{vap} + \frac{5N}{2} (\tau_{in} - \tau_{liq})} \]

part b. The liquefaction coefficient \( \lambda \) is plotted in Fig. 1. It looks qualitatively similar to the corresponding plot in Fig. 12.4 in the text. The two plots match well at low pressures, but become more different at higher pressures.
Problem 2. The pump draws in helium at the vapor pressure of the liquid, and at room temperature. The specific volume of the helium under these conditions, as a function of its pressure $p$, is given by the ideal gas law,

$$V = \frac{\tau_{room}}{p} = \frac{\tau_{room}}{760\text{torr}} = \frac{24\text{L}}{760\text{torr}} \text{mol} \times \frac{p}{760\text{torr}}.$$  

The rate at which gas needs to be removed, in moles per second, is found by dividing the load power by the latent heat. Multiplying by the specific volume, we find the rate (in liters per second) at which the helium must be pumped out,

$$S_{pump} = \frac{P_{load} V}{L_{vap} N} = \frac{P_{load}}{L_{vap}} \times \frac{24\text{L}}{\text{mol}} \times \frac{760\text{torr}}{p}.$$  

The minimum achievable pressure is found from the maximum pump speed,

$$p_{min} = \frac{P_{load}}{S_{pump,max} L_{vap}} \times \frac{24\text{L}}{\text{mol}} \times 760\text{torr}.$$  

In case 1, we have $P_{load} = 0.1\text{W}$ and $S_{pump,max} = 10^3\text{L/sec}$, which gives a minimum pressure of $p_{min} = 0.222\text{torr}$. The text tabulates the pressures and temperatures of helium vapor-liquid equilibrium, so we can use linear interpolation of $T$ vs. $\log_{10}(p)$ and the above minimum pressure to compute the minimum temperature,

$$T_{min} = 0.98\text{K} + \frac{1.27\text{K} - 0.98\text{K}}{\log_{10}(1) - \log_{10}(0.1)} (\log_{10}(0.222) - \log_{10}(0.1)) = 1.08\text{K}.$$  

In case 2, $P_{load} = 0.001\text{W}$ and $S_{pump,max} = 10^3\text{L/sec}$. Using the same procedure as above, we find $p_{min} = 2.22 \times 10^{-4}\text{torr}$ and $T_{min} = 0.595\text{K}$. This shows just how difficult it is to cool helium below about 1K; a thousandfold decrease in vapor pressure yields only a factor of two decrease in temperature.
Problem 3.

\[ |\Delta n| \ll n_i \]

\[ \Rightarrow n_e = \frac{1}{2} \left( \sqrt{4n_i^2 + \Delta n^2} \pm \Delta n \right) = \frac{\sqrt{n_i^2 + \frac{\Delta n^2}{4}} \pm \frac{\Delta n}{2}}{n_i} \]

\[ = n_i \left( 1 + \frac{\Delta n}{4n_i^2} \right) \approx n_i \left( 1 + \frac{\Delta n^2}{8n_i^2} \right) \pm \frac{\Delta n}{2} \]

\[ = n_i \pm \frac{\Delta n}{2} + \frac{\Delta n^2}{8n_i} \]

Problem 4.

part a.

\[ \sigma = e(n_e \tilde{\mu}_e + n_h \tilde{\mu}_h) \]

\[ = \frac{e\tilde{\mu}_e}{2} \left( \sqrt{\Delta n^2 + 4n_i^2} + \Delta n \right) + \frac{e\tilde{\mu}_h}{2} \left( \sqrt{\Delta n^2 + 4n_i^2} - \Delta n \right) \]

\[ = \frac{e(\tilde{\mu}_e + \tilde{\mu}_h)}{2} \sqrt{\Delta n^2 + 4n_i^2} + \frac{e(\tilde{\mu}_e - \tilde{\mu}_h)}{2} \Delta n \]

\[ \Rightarrow 0 = \frac{d\sigma}{d\Delta n} = \frac{e(\tilde{\mu}_e - \tilde{\mu}_h)}{2} \frac{\Delta n}{\sqrt{\Delta n^2 + 4n_i^2}} \]

\[ \Rightarrow \Delta n^2(\tilde{\mu}_e^2 + \tilde{\mu}_h^2 + 2\tilde{\mu}_e\tilde{\mu}_h) = (\Delta n^2 + 4n_i^2)(\tilde{\mu}_e^2 + \tilde{\mu}_h^2 - 2\tilde{\mu}_e\tilde{\mu}_h) \]

\[ = \Delta n^2(\tilde{\mu}_e^2 + \tilde{\mu}_h^2 - 2\tilde{\mu}_e\tilde{\mu}_h + 4n_i^2(\tilde{\mu}_e - \tilde{\mu}_h)^2) \]

\[ \Rightarrow 4\tilde{\mu}_e\tilde{\mu}_h\Delta n^2 = 4n_i^2(\tilde{\mu}_e - \tilde{\mu}_h)^2 \]

\[ \Rightarrow \Delta n^2 = n_i^2 \frac{\tilde{\mu}_e - \tilde{\mu}_h}{\tilde{\mu}_e\tilde{\mu}_h} \Rightarrow \Delta n = \pm n_i \frac{\tilde{\mu}_e - \tilde{\mu}_h}{\sqrt{\tilde{\mu}_e\tilde{\mu}_h}} \]

\[ \Rightarrow \sigma_{\text{min}} = \frac{en_i}{2} \left( \frac{\tilde{\mu}_e + \tilde{\mu}_h}{\sqrt{\tilde{\mu}_e\tilde{\mu}_h}} \right) \left( \frac{(\tilde{\mu}_e - \tilde{\mu}_h)^2}{\tilde{\mu}_e\tilde{\mu}_h} + 4 \pm \frac{(\tilde{\mu}_e - \tilde{\mu}_h)(\tilde{\mu}_e - \tilde{\mu}_h)}{\sqrt{\tilde{\mu}_e\tilde{\mu}_h}} \right) \]

Picking the negative sign in the parenthesis in order to minimize \( \sigma \), we obtain

\[ \Delta n = -n_i \frac{\tilde{\mu}_e - \tilde{\mu}_h}{\sqrt{\tilde{\mu}_e\tilde{\mu}_h}}, \text{ and } \sigma_{\text{min}} = 2en_i \sqrt{\tilde{\mu}_e\tilde{\mu}_h}. \]

part b.

\[ \sigma_{\text{int}} = en_i(\tilde{\mu}_e + \tilde{\mu}_h) \]

\[ \Rightarrow \frac{\sigma_{\text{min}}}{\sigma_{\text{int}}} = \frac{2\sqrt{\tilde{\mu}_e\tilde{\mu}_h}}{\tilde{\mu}_e + \tilde{\mu}_h} \]

part c. For Si, \( \sigma_{\text{min}} = 1.19 \times 10^{-4} \Omega^{-1} m^{-1} \) and for Sb, \( \sigma_{\text{min}} = 4.006 \times 10^{3} \Omega^{-1} m^{-1} \).

Problem 5.
Problem 6.

\[ n_{d^+} = n_e = \frac{1}{4} n_{e^+} \left( \sqrt{1 + \left( \frac{8n_d}{n_e^2} \right)} - 1 \right) \]

with \( n_{e^*} = n_e e^{-\Delta \epsilon_d / \tau} \Rightarrow \frac{8n_d}{n_{e^*}} = \frac{8n_d}{n_c} e^{\Delta \epsilon_d / \tau} \)

To first order,

\[ n_e \approx \frac{1}{4} n_{e^*} \sqrt{\frac{8n_d}{n_e} e^{\Delta \epsilon_d / \tau}} = \frac{1}{4} n_e e^{-\Delta \epsilon_d / \tau} \sqrt{\frac{8n_d}{n_c} e^{\Delta \epsilon_d / \tau}} = \sqrt{\frac{n_c n_d e^{-\Delta \epsilon_d / (2 \tau)}}{2 n_d}}. \]

Problem 7.

\[ n_a(x) = n_h e^{\phi / \tau} \]

\[ \Rightarrow \frac{n_a(x_1)}{n_a(x_2)} = \frac{n_1}{n_2} = e^{\phi_1 / \tau} e^{\phi_2 / \tau} = e^{(\phi_1 - \phi_2) / \tau} \]

\[ \Rightarrow \phi_1 - \phi_2 = \frac{\tau}{e} \log \left( \frac{n_1}{n_2} \right) \]

\[ \Rightarrow E = -\nabla \phi \approx -\frac{\phi_1 - \phi_2}{x_1 - x_2} = \frac{\tau \log(n_1/n_2)}{e(x_2 - x_1)} = 1.78 \times 10^6 \text{V/m} \]