Problem set 2 solutions

Problem 1.

part a.
- most probable velocity:

\[
0 = \frac{dp(v)}{dv} = 4\pi \left( \frac{m}{2\pi \tau} \right)^{3/2} \left( 2\nu e^{-\frac{m\nu}{2\tau}} - v^2 \left( \frac{mv}{\tau} \right) e^{-\frac{mv^2}{2\tau}} \right)
\]

\[\Rightarrow 2\nu - \frac{m}{\tau}v^3 = 0 \Rightarrow v^2 = \frac{2\tau}{m} \]

\[\Rightarrow v_{mp} = \sqrt{\frac{2\tau}{m}} \]

- average velocity:

\[
\langle v \rangle = \int_0^\infty 4\pi \left( \frac{m}{2\pi \tau} \right)^{3/2} v^3 e^{-\frac{mv^2}{2\tau}} dv
\]

\[= \frac{2}{\sqrt{\pi}} \sqrt{\frac{2\tau}{m}} \int_0^\infty x^3 e^{-x^2} dx \text{ (with } x = mv^2/(2\tau)) \]

\[\Rightarrow \langle v \rangle = \frac{2}{\sqrt{\pi}} \sqrt{\frac{2\tau}{m}} \sqrt{\frac{8\tau}{\pi m}} \]

- root mean squared velocity:

\[
\langle v^2 \rangle = \int_0^\infty v^2 p(v) dv = \int_0^\infty 4\pi \left( \frac{m}{2\pi \tau} \right)^{3/2} v^4 e^{-\frac{mv^2}{2\tau}} dv
\]

\[= \frac{2}{\sqrt{\pi}} \frac{2\tau}{m} \int_0^\infty x^3 e^{-x^2} dx \text{ (with } x = mv^2/(2\tau)) \]

\[= \frac{2}{\sqrt{\pi}} \frac{2\tau}{m} \Gamma \left( \frac{5}{2} \right) \]

\[\Rightarrow \nu_{RMS} = \sqrt{\langle v^2 \rangle} = \left( \frac{4}{\pi} \right)^{1/4} \sqrt{\Gamma \left( \frac{5}{2} \right) \frac{2\tau}{m}} = \sqrt{\frac{3\tau}{\pi m}} \]

part b. \( v_{mp} = 417 \text{ m/s}, \langle v \rangle = 471 \text{ m/s}, \text{ and } \nu_{RMS} = 511 \text{ m/s}. \)

Problem 2. Assume that the box has volume \( V_{box} \), the hole has area \( A \), and that the particles are allowed to escape through it during a time period \( \Delta t \).

- The probability of the particle having a velocity between \( v \) and \( v + \Delta v \) is \( p(v)\Delta v \), that is, the Maxwell distribution.
- The probability that the particle's position vector makes an angle between \( \theta \) and \( \theta + \Delta \theta \) with the normal vector to the wall is \( \sin(\theta)\Delta \theta \).
- The probability of having a distance from the hole between \( x \) and \( x + \Delta x \) is \( 2\pi x^2 \Delta x/V_{box} \). Note that only particles with \( x \leq \nu\Delta t \) are close enough to the hole to be able to escape.
- The probability that the particle of velocity \( v \), distance \( x \), and angle \( \theta \) will be traveling in the right direction to reach the hole is \( \frac{A \cos(\theta)}{4\pi x^2} \).
Multiplying all of the above, and including a normalization constant \( C \), we have the probability density corresponding to a particle of velocity \( v \), at a distance \( x \) and angle \( \theta \),

\[
P(v, x, \theta) d\theta dx dv = C p(v) \sin(\theta) \frac{2\pi x^2}{V_{box}} A \cos(\theta) d\theta dx dv
\]

\[
= C p(v) \sin(\theta) \frac{A \cos(\theta)}{2V_{box}} d\theta dx dv.
\]

In order to obtain \( P(v) dv \), we integrate over \( \theta \) between 0 and \( \pi/2 \), and over \( x \) between 0 and \( v\Delta t \).

\[
P(v) dv = \int_0^{v\Delta t} \int_0^{\pi/2} C p(v) \sin(\theta) \frac{A \cos(\theta)}{2V_{box}} d\theta dx dv
\]

\[
= \int_0^{v\Delta t} C p(v) \frac{A}{4V_{box}} x dx dv
\]

\[
= C p(v) \frac{Av\Delta t}{4V_{box}} dv.
\]

Finally, we normalize to find \( C \).

\[
1 = C \frac{A \Delta t}{4V_{box}} \int_0^\infty v p(v) dv = C \frac{A \Delta t}{4V_{box}} \sqrt{\frac{8\pi}{\pi m}}
\]

\[
\Rightarrow P(v) dv = \sqrt{\frac{2m}{8\pi}} v p(v) dv = \frac{m^2}{2\pi^2} v^3 e^{-\frac{m^2}{2\pi v^2}}
\]

**Problem 3.** From lecture 6, the partition function for a single dipole is

\[
Z(\tau) = e^{E/\tau} + e^{-E/\tau} = 2 \cosh (E/\tau).
\]

The free energy of an \( N \) dipole system is \( N \) times the free energy of a single dipole, since they are isolated systems.

\[
F = -N\tau \log(Z) = -N\tau (\log(\cosh(E/\tau)) + \log(2))
\]

\[
\Rightarrow U = -\tau^2 \frac{\partial F}{\partial \tau} = -\tau^2 \frac{\partial}{\partial \tau} (-N \log(\cosh(E/\tau))) = -NE \tanh(E/\tau)
\]

\[
\Rightarrow \sigma = -\left( \frac{\partial F}{\partial \tau} \right)_V = N \left[ \log(\cosh(E/\tau)) + \log(2) \right] - \frac{NE}{\tau} \tanh(E/\tau)
\]

**Problem 4.** The single-system partition functions for systems 1 and 2 are

\[
Z_1(\tau) = \sum_{s_1} e^{-E_{s_1}/\tau} \text{ and } Z_2(\tau) = \sum_{s_2} e^{-E_{s_2}/\tau}.
\]

The partition function for the combined system is

\[
Z(\tau) = \sum_{s_1} \sum_{s_2} e^{-(E_{s_1} + E_{s_2})/\tau} = \sum_{s_1} \sum_{s_2} e^{-E_{s_1}/\tau} e^{-E_{s_2}/\tau} = \sum_{s_1} \left[ e^{-E_{s_1}/\tau} \sum_{s_2} e^{-E_{s_2}/\tau} \right]
\]

\[
= \sum_{s_1} \left[ e^{-E_{s_1}/\tau} Z_2(\tau) \right] = Z_1(\tau) \sum_{s_1} e^{-E_{s_1}/\tau} = Z_1(\tau) Z_2(\tau)
\]

**Problem 5.**
part a.

\[ Z(\tau) = \sum_{s=0}^{N} e^{-s\epsilon/\tau} \] (where \( s \) links are open in state \( s \))

\[ \Rightarrow e^{-\tau/\tau} Z = \sum_{s=1}^{N+1} e^{-s\epsilon/\tau} \]

\[ \Rightarrow Z - e^{-\epsilon/\tau} Z = 1 - e^{-(N+1)\epsilon/\tau} \]

\[ \Rightarrow Z = \frac{1 - e^{-(N+1)\epsilon/\tau}}{1 - e^{-\epsilon/\tau}} \]

part b.

\[ \langle s \rangle = \frac{1}{Z} \sum_{s=0}^{N} g e^{-s\epsilon/\tau} = \frac{1}{Z} e^{\frac{\tau^2}{\epsilon}} \frac{dZ}{d\tau} \]

\[ = \frac{1 - e^{-\epsilon/\tau}}{1 - e^{-(N+1)\epsilon/\tau}} \left[ \frac{1 - e^{-(N+1)\epsilon/\tau}}{(1 - e^{-\epsilon/\tau})^2} e^{-\epsilon/\tau} - \frac{e^{-(N+1)\epsilon/\tau} (N + 1)}{1 - e^{-\epsilon/\tau}} \right] \]

\[ = \frac{e^{-\epsilon/\tau}}{1 - e^{-\epsilon/\tau}} - \frac{(N + 1)e^{-(N+1)\epsilon/\tau}}{1 - e^{-(N+1)\epsilon/\tau}} \]

As \( \epsilon/\tau \) becomes large, \( e^{-(N+1)\epsilon/\tau} \) decreases much more quickly than \( e^{-\epsilon/\tau} \), so the second term above quickly vanishes. Thus we're left with

\[ \langle s \rangle \approx \frac{e^{-\epsilon/\tau}}{1 - e^{-\epsilon/\tau}} \approx e^{-\epsilon/\tau}. \]

Problem 6.

part a. The excess of right-directed links, \( 2s \), is analogous to the spin excess from the paramagnetic spin system discussed in class. The number of right-directed links is \( n_r = \frac{1}{2} N + s \), so the number of states with a given \( s \) is

\[ g(N, s) = \frac{N!}{(N - n_r)! n_r!} = \frac{N!}{(\frac{1}{2} N + s)! (\frac{1}{2} N - s)!}. \]

Changing the sign of \( s \) doesn’t change \( g \), so we have

\[ g(N, s) + g(N, -s) = \frac{2N!}{(\frac{1}{2} N + s)! (\frac{1}{2} N - s)!}. \]

part b.

\[ g(N, s) + g(N, -s) = \frac{2N!}{(\frac{1}{2} N)! (\frac{1}{2} N)!} \frac{(\frac{1}{2} N)! (\frac{1}{2} N)!}{(\frac{1}{2} N + s)! (\frac{1}{2} N - s)!} \]

\[ = 2g(N, 0) \frac{(\frac{1}{2} N)! (\frac{1}{2} N)!}{(\frac{1}{2} N + s)! (\frac{1}{2} N - s)!} \]
\[
\Rightarrow \sigma = \log(g(N, s) + g(N, -s)) = \log(2g(N, 0)) + \log \left[ \frac{(\frac{1}{2}N)!}{(\frac{1}{2}N)!} \left( \frac{1}{2}N + s \right)! \left( \frac{1}{2}N - s \right)! \right] \\
\approx \log(2g(N, 0)) + 2 \left[ \frac{1}{2}N \log \left( \frac{N}{2} \right) - \frac{N}{2} \right] - \left[ \left( \frac{1}{2}N + s \right) \log \left( \frac{1}{2}N + s \right) - \left( \frac{1}{2}N - s \right) \right] \\
- \left[ \left( \frac{1}{2}N + s \right) \log \left( \frac{1}{2}N + s \right) - \left( \frac{1}{2}N + s \right) \right] \\
= \log(2g(N, 0)) - \frac{N}{2} \log \left( \frac{N - 2s}{N} \right) - \frac{N}{2} \log \left( \frac{N + 2s}{N} \right) + s \log \left( \frac{N}{2} - s \right) - s \log \left( \frac{N}{2} + s \right) \\
\approx \log(2g(N, 0)) - \frac{N}{2} \left( \frac{2s}{N} \right)^2 - \frac{N}{2} \left( 2s - \frac{1}{2} \left( \frac{2s}{N} \right)^2 \right) + s \left( \frac{2s}{N} \right) - s \left( \frac{2s}{N} \right) \\
= \log(2g(N, 0)) - \frac{2s^2}{N} = \log(2g(N, 0)) - \frac{f^2}{2N\rho^2} \\
\]

\text{part c.} \\
\begin{align*}
-\frac{f}{\tau} &= \left( \frac{\partial \sigma}{\partial f} \right)_U = -\frac{2\ell}{N\rho^2} \Rightarrow f &= \frac{\ell\tau}{N\rho^2} \\
\end{align*}

\text{part d. Expanding a rubber band makes it \textbf{hotter}.}

\textbf{Problem 7.} For a single particle in a box, the partition function is given by
\[
Z_1 = \sum_{n} \exp \left( \frac{-\pi^2 h^2 n_x^2}{2ML^2r} \right) \approx \int_{0}^{\infty} \exp \left( \frac{-\pi^2 h^2 n_x^2}{2ML^2r} \right) dn_x = \sqrt{\frac{2ML^2r}{\pi^2 h^2}} \int_{0}^{\infty} e^{-x^2} dx \\
= \sqrt{\frac{2ML^2r}{\pi^2 h^2}} \sqrt{\pi} = \sqrt{\frac{ML^2r}{2\pi h^2}}. \\
\]
A system of \(N\) indistinguishable particles has the partition function
\[
Z_N = \frac{1}{N!} Z_1^N = \frac{1}{N!} \left( \frac{ML^2r}{2\pi h^2} \right)^{N/2}. \\
\]
\[
\Rightarrow F = -\tau \log(Z_N) = -\tau \left[ N \log \sqrt{\frac{ML^2r}{2\pi h^2}} + N - N \log(N) \right] = -\tau N \left( \log(n_Q/n) + 1 \right) \\
\text{(where } n = N/L \text{ and } n_Q = \sqrt{M/2\pi h^2}) \\
\Rightarrow \sigma = -\left( \frac{\partial F}{\partial \tau} \right)_L = \frac{\partial}{\partial \tau} [\tau N \left( \log(n_Q/n) + 1 \right)] = N \left( \log(n_Q/n) + 1 \right) + \tau N \frac{1}{2} \\
= N \left( \log(n_Q/n) + \frac{3}{2} \right) = N \left[ \log \left( \frac{L}{N} \frac{\sqrt{M\tau}}{2\pi h^2} \right) + \frac{3}{2} \right].
\]
Problem 8. Assume that the probabilities $p_i$ are each varied by a small amount $\delta p_i$, subject to the constraint that $\sum_i \delta p_i = 0$. $\sigma$ and $U$ vary as follows.

$$\delta \sigma = \sum_i \frac{\partial \sigma}{\partial p_i} \delta p_i = - \sum_i (\log(p_i) + 1) \delta p_i = \sum_i (1 + \lambda_1 + \lambda_2 E_i - 1) \delta p_i$$

$$= \lambda_1 \sum_i \delta p_i + \lambda_2 \sum_i E_i \delta p_i = \lambda_2 \sum_i E_i \delta p_i$$

$$\delta U = \sum_i \frac{\partial U}{\partial p_i} \delta p_i = \sum_i E_i \delta p_i$$

From the definition of temperature, we have

$$\frac{1}{\tau} = \frac{\delta \sigma}{\delta U} = \frac{\lambda_2 \sum_i E_i \delta p_i}{\sum_i E_i \delta p_i} = \lambda_2 \Rightarrow \lambda_2 = \frac{1}{\tau}.$$