GPS in Flat Land

Introductory material

The real current GPS constellation consists of 24 active (and three backup) satellites. Four active satellites exist in each of six orbital planes positioned around the earth at an altitude of 20,200 km (to enable 2 orbits a day). These planes are positioned at 55º intervals to the equator. As the earth is an oblate spheroid, real satellite orbits are elliptical rather than circular.

A real almanac would include satellite ID #, time of almanac and √a (note that in the real context, the value of a would refer to the semi-major axis). Its other columns, however, would consist of health, eccentricity, orbital inclination, right ascension at time of almanac, rate of right ascension, argument of perigee, mean anomaly, Af (0), Af (1) and week. Orbital inclination, right ascent at time of almanac and rate of right ascension are the only ones that are not applicable to flat land. The others are not included to simplify the problem.

As the GPS system is based on weeks and was only constructed to work up to week 1023, week 1024 presented a problem akin to the Y2K problem. It occurred on August 22, 1999. The next W1024 problem will occur in week 2048 (1024 weeks after this date) on March 5, 2019.

These steps outline my approach to calculate the precise position of the handheld receiver:

Step One

My first step was to use calculate the longitudes, L₁, of the three transmitting satellites based on three values already known to us: the time of transmission, t₁, the longitude, L₀, at t = 0, and the period, p, of each satellite. The last can be taken to be half a sidereal day or calculated by Kepler’s Third Law (as we know both the satellite’s radius and what GM is):

\[ L_1 = (t_1 \times (360/p)) + L_0 \]
The three values for $L_1$ that I obtained were as follows:

- $253.191597977553^\circ$ for satellite 2
- $298.191626604492^\circ$ for satellite 3
- $343.191531863742^\circ$ for satellite 4

**Step Two**

After I obtained the three $L_1$ values, I was able to obtain an expression for the distances between the receiver and each satellite that was dependant on two variables: altitude of receiver and longitude of receiver. An illustration aids the derivation of this expression as follows:

In the illustration above, the altitude of the receiver is represented by $h$, while $\theta$ reflects $L_1$ – the longitude of the receiver, which we shall call $L_R$. The symbol $r$ refers to the distance from the earth to sea level (which I took to be 6371000m) and $a$ is the radius of the satellite’s orbit (which = 26561819.361856m). I now applied the law of cosines as follows:

$$d = \sqrt{(a^2 + (r+h)^2 - 2a(r+h)\cos\theta)}$$

Note, when I entered this formula into my spreadsheet later, I used values of $\theta$ in radians.
Step Three
From the three expressions for the distances between the receiver and the satellites, I derived expressions for each satellite for the transmission delay and, by consequence, for time of reception, $t_R$. The time of reception is the time of transmission + the transmission delay:

\[
\text{delay} = \frac{\text{transmission distance}}{\text{transmission velocity}} = \frac{d}{c} \quad (c = 299792458)
\]

\[
t_R = t_T + \frac{d}{c}
\]

Step Four
The next step consisted of a process of trial and improvement in which I used a spreadsheet to vary values of $L_R$ and $h$ in order to obtain the same $t_R$ value for each receiver. Initially, I let $h = 0$ and varied values of $L_R$. The values of $t_T$ revealed that the position of the receiver is closest to satellite 3 and furthest from satellite 4. This means that $L_R$ must fall in the interval $275^\circ < L_R < 291^\circ$. Within this interval, I was able to narrow down the value $L_R$ to between $285.238^\circ$ and $285.239^\circ$ keeping $h$ at 0. I now varied $h$ simultaneously in order to attain receiver time, altitude and longitude to the desired accuracy. The results of this process are shown on the spreadsheet attached with this report.

Results

<table>
<thead>
<tr>
<th>Time (to 0.000000001 seconds):</th>
<th>22222.221136001 seconds</th>
</tr>
</thead>
<tbody>
<tr>
<td>Longitude (to 0.000001 degrees):</td>
<td>285.238719º</td>
</tr>
<tr>
<td>Altitude (to 1 meter):</td>
<td>6371061m (i.e. 61m above sea level)</td>
</tr>
</tbody>
</table>

Follow-up material
If we were to consider the rotation of the earth, the satellites will complete two rotations for every one rotation of the earth. This would make the satellites appear to be moving half as slowly as they would have been when the earth was stationary to an observer on earth.